Checking in linear time if an S-term normalizes

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Abstract

Curry's Combinatory Logic is a functional calculus which may serve as a foundation to the theory of computations, even to computational complexity. Combinatory Logic, which is based on the two combinators S and K, is an undecidable theory. The theory based only on S was proved decidable by Waldmann. We present a regular tree grammar and prove that it exactly produces all normalizing S-terms. Thus, the complexity of deciding whether an S-term X has a normal form is linear. Our grammar is equivalent to another one by Waldmann, but in contrast to our grammar, the correctness of Waldmann's grammar was only proved with the help of computer programs.

1 Introduction

We call S-terms the elements of a system generated by one symbol S and one non-associative and non-commutative (implicit) operation that we call application:

(i) S is an S-term.

(ii) If m_1 and m_2 are S-terms, $(m_1 m_2)$ is an S-term.

In this construction we say that m_1 and m_2 are *proper sub-terms* of the constructed S-term. S is a sub-term of any S-term. Also, we say that an S-term is a sub-term of itself. We may abbreviate by omitting parentheses by using left association. For example, we write $SS(S(SSS))S$ instead of $(((SS)(S((SS)S)))S)$ and $xyz = (xy)z \neq x(yz)$.

Here, we use lower case italic letters to represent S-terms. We use upper case calligraphic letters to represent sets of S-terms. For any sets of S-terms $\mathcal{A}$ and $\mathcal{C}$, we will write $\mathcal{AC} = \{\, ac \mid a \in \mathcal{A} \text{ and } c \in \mathcal{C} \,\}$.

We define the length of an S-term to be the number of occurrences of the symbol S in the term. For any S-term x, we write $|x|$ to denote the length of x.

The reduction relation $\rightarrow$ is defined here by the S-rule:

$$Sxyz \xrightarrow{\text{def}} xz(yz)\,.$$

The left hand side, $Sxyz$, is sometimes called *redex* and the right hand side, $xz(yz)$, *reductum*. For example, $SSdc \longrightarrow Sc(dc)$. In general we write $x \rightarrow y$ if y can be written by replacing some redex, sub-term, in x by the corresponding reductum of the S-rule.

The original motivation of this problem was the need to create a *Functional Calculus* instead of *Set Theory* as a foundation for Theory of Computation, i.e., for Computability (Thue, Schönfinkel [12], Curry [6, 7], Church [4], Turing, Markov) but even for Computational Complexity. In such a functional calculus only *one operation* is needed: application $f(g)$. We write (fg) instead of $f(g)$. Schönfinkel made the following observation: functions of one argument are enough, e.g., $f(g,h) = ((fg)h)$. We use left association for dropping some parentheses, i.e., instead of $((((fg)(h(gh)))((gh)((fh)f))))$ we write $fg(h(gh))(gh(fhf))$. Many people have been involved in similar investigations [17, 1, 2, 3, 8, 10, 11, 13, 15, 14, 9].

An example is the SK-Calculus or Combinatory Logic of H. Curry [6] which contains the system with S-terms and the S-rule. It has the rules $St_1t_2t_3 \rightarrow t_1t_3(t_2t_3)$ and $Kt_1t_2 \rightarrow t_1$. Actually there are many other combinators and rewriting rules but the system $\{S, K\}$ is complete.

When reducing by the S-rule we eliminate one symbol S from an S-term and introduce a replica of a sub-term in the S-term, hence, if $x \to y$, $|x| \leq |y|$. Thus a reduction step certainly does not reduce the length of the S-term.

We use the abbreviations $A \stackrel{\text{def}}{=} SSS$ and $B \stackrel{\text{def}}{=} S(SS)$. Applying the S-rule twice we get:

$$Bad = S(SS)ad \longrightarrow SSd(ad) \longrightarrow S(ad)(d(ad)),$$

which we will write:

$$Bad \xrightarrow{2} S(ad)(d(ad)).$$

In general, for $k \geq 0$, we write $\xrightarrow{k}$ to represent k reduction steps.

Here we describe other extensions of the relation $\to$. The transitive closure of $\to$ is denoted by $\xrightarrow{+}$ and its reflexive transitive closure is denoted by $\xrightarrow{*}$. For two sets of S-terms $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \to \mathcal{Y}$ if for any $x \in \mathcal{X}$ we can apply the S-rule on some redex sub-term of x so that $x \to y$ for some $y \in \mathcal{Y}$. Similarly, we will extend the other relations described above to sets of S-terms.

We say that an S-term x *is in normal form* if the S-rule cannot be applied to any sub-term of x, i.e., there is no redex in x. We say that x *has a normal form* and write $x\downarrow$ if $x \xrightarrow{*} n$ for some n in normal form; we write $x\uparrow$ otherwise, i.e., if x *does not have a normal form*, which is equivalent to: there is a non-terminating reduction chain starting with x.

The question whether a given S-term is normalizable was answered positively in [16]:

THEOREM 1. *There is an algorithm that decides if a given S-term has a normal form.*

A *regular tree grammar* is a tuple $G = (I, N, F, R)$, where I is the axiom, N is the set of non-terminal symbols (with $I \in N$), F is the set of terminal symbols, and R is the set of rules. Moreover, each rule is of the form $A ::= \beta$, where $A \in N$ and β is a tree on $N \cup F$. Also, the arity of all non-terminal symbols is 0. For more details and for the set of terms generated by a regular tree grammar, see [5].

Waldmann [16] showed that the set of normalizable terms can be generated by a regular tree grammar, but his proof relies on a computer program. In this paper, we give a regular tree grammar generating exactly all normalizable S-terms and we provide a proof of its correctness without a computer. As a result, since one can construct a deterministic tree automaton from a regular tree grammar [5], it is possible to decide normalization of an S-term in linear time.

2 Notations

We first introduce some further notation.

Suppose $x \to y$. Then, for any sub-term z of y we will write $x \xrightarrow{\circ} z$. For example:

$$Sadc \xrightarrow{\circ} dc.$$

As extensions of $\xrightarrow{\circ}$, we will denote its transitive closure by $\xrightarrow{\oplus}$ and its reflexive transitive closure by $\xrightarrow{\circledast}$. Using this notation we have the following fact:

> Suppose $\mathcal{X} \xrightarrow{\oplus} \mathcal{X}$. Then, there is an infinite reduction chain starting with any $x \in \mathcal{X}$, i.e., $\mathcal{X}\uparrow$.

We are using a notation similar to that of regular expressions, e.g., we write S instead of $\{S\}$, we write $\mathcal{X}+\mathcal{Y}$ instead of $\mathcal{X}\cup\mathcal{Y}$, etc. We define the sets $\mathcal{M}$ and $\mathcal{N}$ to be the (least) solutions of the fixed point equations:

$$\begin{aligned} \mathcal{M} &\stackrel{\text{def}}{=} S+\mathcal{M}\mathcal{M}\,, &&\text{and}\\ \mathcal{N} &\stackrel{\text{def}}{=} S+S\mathcal{N}+S\mathcal{N}\mathcal{N}\,. \end{aligned}$$

That is, $\mathcal{M}$ is the set of all S-terms; $\mathcal{N}$ is the set of all S-terms that are in normal form. For any set $\mathcal{X}$ we define $\overline{\mathcal{X}}=\mathcal{M}-\mathcal{X}$.

With this notation, we will also define the sets:

$$\begin{aligned} \mathcal{Q}_1 &\stackrel{\text{def}}{=} \overline{S}\,, \\ \mathcal{Q}_2 &\stackrel{\text{def}}{=} \overline{S+SS}\,, &&\text{and}\\ \mathcal{Q}_3 &\stackrel{\text{def}}{=} \overline{S+SS+S(SS)} \;=\; \overline{S+SS+B}\,. \end{aligned}$$

So $\mathcal{Q}_1$ is the set of all S-terms of length greater than one; $\mathcal{Q}_2$ is the set of all S-terms of length greater than two. Some immediate facts are:

$$\begin{aligned} \mathcal{Q}_1 &= SS+\mathcal{Q}_2\,, \\ \mathcal{M}\mathcal{Q}_i &\subseteq \mathcal{Q}_{i+1} \subseteq \mathcal{Q}_i \qquad \text{for } i=1 \text{ and } i=2, \text{ and}\\ \mathcal{M}\mathcal{Q}_3 &\subseteq \mathcal{Q}_3\,. \end{aligned}$$

Since every redectum is in $\mathcal{M}\mathcal{M}\mathcal{M}\subseteq\mathcal{Q}_3\subseteq\mathcal{Q}_2\subseteq\mathcal{Q}_1$, we can always write $x\to\mathcal{Q}_i$ for any redex x (or any term x that has a redex!) and $i=1,2,$ and 3.

For sets of S-terms $\mathcal{A}$ (the *prefix* set) and $\mathcal{D}$ (the *base* set) we recursively define $(\mathcal{A})^n[\mathcal{D}]$ for all $n\geq 0$ with:

$$\begin{aligned} (\mathcal{A})^0[\mathcal{D}] &= \mathcal{D} &&\text{and}\\ (\mathcal{A})^{k+1}[\mathcal{D}] &= \mathcal{A}\big((\mathcal{A})^k[\mathcal{D}]\big) &&\text{for } k\geq 0\,. \end{aligned}$$

The set of all terms defined above is:

$$(\mathcal{A})^*[\mathcal{D}] \;=\; \sum_{n\geq 0}(\mathcal{A})^n[\mathcal{D}] \;=\; (\mathcal{A})^0[\mathcal{D}]+(\mathcal{A})^1[\mathcal{D}]+(\mathcal{A})^2[\mathcal{D}]+\cdots$$

which is the (least) solution of the fixed point equation:

$$(\mathcal{A})^*[\mathcal{D}] \;=\; \mathcal{D}\,+\,\mathcal{A}\big((\mathcal{A})^*[\mathcal{D}]\big)\,.$$

Remark 1. The expressions defined above describe sets of normal forms when the prefix $\mathcal{A}\subseteq S+S\mathcal{N}$ and the base $\mathcal{D}\subseteq\mathcal{N}$, e.g., $(S\mathcal{N})^1[\mathcal{N}]=S\mathcal{N}\mathcal{N}\subseteq\mathcal{N}$.

Example:

$$\begin{aligned} &(SS+B)^*\,[S\mathcal{N}] \;=\\ &\quad \mathcal{N}+SS(S\mathcal{N})+B(S\mathcal{N})+SS(SS(S\mathcal{N}))+SS(B(S\mathcal{N}))+B(SS(S\mathcal{N}))+B(B(S\mathcal{N}))+\cdots \end{aligned}$$

Remark 2. Whenever $(\mathcal{A})^*[\mathcal{D}]$ is defined so that $\mathcal{D}$ has no term also belonging to $\mathcal{A}\mathcal{M}$, if $ax\in(\mathcal{A})^*[\mathcal{D}]$ for some $a\in\mathcal{A}$, $x\in(\mathcal{A})^*[\mathcal{D}]$.

Using this notation we define:

$$\mathcal{E} \stackrel{\text{def}}{=} (SS)^*[\mathcal{Q}_2\mathcal{Q}_1]\,.$$

Remark 3. $\mathcal{Q}_1\mathcal{E}=(SS+\mathcal{Q}_2)\mathcal{E}=SS\mathcal{E}+\mathcal{Q}_2\mathcal{E}\subseteq\mathcal{E}+\mathcal{Q}_2\mathcal{Q}_1\subseteq\mathcal{E}$, i.e., for any $q\in\mathcal{Q}_1$, $q\mathcal{E}\subseteq\mathcal{E}$. That way, we can say $\mathcal{E}=(\mathcal{Q}_1)^*[\mathcal{Q}_2\mathcal{Q}_1]$.

3 The grammar

The regular tree grammar has the axiom non-terminal symbol $\langle\mathcal{N}\rangle$ (which generates all normalizable S-terms). It has the following rules. The proof of the correctness of the grammar is quite lengthy and is given in the appendix.

$$
\begin{aligned}
\langle\mathcal{N}\rangle &::= S \mid \langle\mathcal{L}_0\rangle\,\langle\mathcal{N}\rangle \mid \langle\mathcal{H}_0\rangle\,\langle\mathcal{L}_0\rangle \mid \langle\mathcal{L}_1\rangle\,\langle\mathcal{H}_0\rangle \mid \langle\mathcal{L}_2\rangle\,\langle\mathcal{L}_1\rangle \mid \langle\mathcal{N}^{-S}\rangle\,S \mid \langle\mathcal{L}_4\rangle\,(\,S\,S\,)\\
\langle\mathcal{H}_0\rangle &::= S \mid S\,\langle\mathcal{N}\rangle \mid S\,S\,\langle\mathcal{H}_0\rangle \mid B\,\langle\mathcal{H}_0\rangle \mid \langle\mathcal{H}_0^{-S}\rangle\,S \mid \langle\mathcal{H}_0^{-SS}\rangle\,(\,S\,S\,)\\
\langle\mathcal{H}_0^{-S}\rangle &::= \langle\mathcal{H}_0^{-SS}\rangle \mid \langle\mathcal{K}_3\rangle\\
\langle\mathcal{H}_0^{-SS}\rangle &::= \langle\mathcal{L}_0^{-S}\rangle \mid \langle\mathcal{L}_1^{-S}\rangle\\
\langle\mathcal{L}_0^{-S}\rangle &::= \langle\mathcal{K}_0\rangle \mid \langle\mathcal{K}_1\rangle\\
\langle\mathcal{L}_1^{-S}\rangle &::= B \mid S\,B \mid S\,S\,\langle\mathcal{L}_1^{-S}\rangle\\
\langle\mathcal{K}_3\rangle &::= \langle\mathcal{K}_2\rangle\,S \mid S\,S\,\langle\mathcal{K}_3\rangle\\
\langle\mathcal{K}_2\rangle &::= \langle\mathcal{K}_1\rangle \mid S\,\langle\mathcal{L}_0^{-S}\rangle \mid S\,S\,\langle\mathcal{K}_2\rangle\\
\langle\mathcal{K}_1\rangle &::= \langle\mathcal{K}_0\rangle\,S \mid S\,S\,\langle\mathcal{K}_1\rangle\\
\langle\mathcal{K}_0\rangle &::= S \mid \langle\mathcal{K}_1\rangle\,S \mid S\,S\,\langle\mathcal{K}_0\rangle\\
A &::= S\,S\,S\\
B &::= S\,(\,S\,S\,)\\
\langle\mathcal{L}_0\rangle &::= S \mid S\,\langle\mathcal{N}\rangle \mid \langle\mathcal{L}_0^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_0\rangle\\
\langle\mathcal{L}_1\rangle &::= \langle\mathcal{L}_1^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_1\rangle\\
\langle\mathcal{L}_2\rangle &::= B\,(\,S\,S\,) \mid A\,(\,S\,S\,) \mid B\,B \mid \langle\mathcal{L}_2^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_2\rangle\\
\langle\mathcal{L}_2^{-S}\rangle &::= \langle\mathcal{K}_4\rangle\,S \mid S\,S\,\langle\mathcal{L}_2^{-S}\rangle\\
\langle\mathcal{K}_4\rangle &::= B \mid S\,S\,\langle\mathcal{K}_4\rangle\\
\langle\mathcal{L}_4\rangle &::= S\,S\,\langle\mathcal{L}_4\rangle \mid B\,\langle\mathcal{L}_4\rangle \mid \langle\mathcal{L}_4^{-S}\rangle\,S \mid \langle\mathcal{L}_4^{-SS}\rangle\,(\,S\,S\,)\\
\langle\mathcal{L}_4^{-S}\rangle &::= \langle\mathcal{L}_4^{-SS}\rangle \mid \langle\mathcal{K}_6\rangle\\
\langle\mathcal{L}_4^{-SS}\rangle &::= \langle\mathcal{K}_0\rangle\,S \mid S\,\langle\mathcal{H}_0^{-SS}\rangle \mid S\,S\,\langle\mathcal{L}_4^{-SS}\rangle\\
\langle\mathcal{K}_6\rangle &::= \langle\mathcal{K}_5\rangle\,S \mid S\,S\,\langle\mathcal{K}_6\rangle\\
\langle\mathcal{K}_5\rangle &::= \langle\mathcal{K}_0\rangle\,S \mid S\,\langle\mathcal{K}_0\rangle \mid S\,S\,\langle\mathcal{K}_5\rangle\\
\langle\mathcal{N}^{-S}\rangle &::= S \mid S\,\langle\mathcal{N}\rangle \mid S\,\langle\mathcal{H}_0^{-S}\rangle\,\langle\mathcal{L}_0^{-S}\rangle \mid S\,\langle\mathcal{L}_1^{-S}\rangle\,\langle\mathcal{H}_0^{-S}\rangle \mid S\,\langle\mathcal{L}_2^{-S}\rangle\,\langle\mathcal{L}_1^{-S}\rangle \mid S\,\langle\mathcal{L}_4^{-S}\rangle\,S \mid\\
&\qquad \langle\mathcal{K}_0\rangle\,\langle\mathcal{L}_0^{-S}\rangle \mid \langle\mathcal{J}_1\rangle\,B \mid \langle\mathcal{J}_2\rangle\,\langle\mathcal{L}_0^{-S}\rangle \mid \langle\mathcal{J}_3\rangle\,S \mid \langle\mathcal{J}_4\rangle\,\langle\mathcal{L}_1^{-S}\rangle \mid \langle\mathcal{L}_1\rangle\,S \mid \langle\mathcal{J}_5\rangle\,(\,S\,S\,) \mid\\
&\qquad \langle\mathcal{J}_6\rangle\,\langle\mathcal{L}_0^{-S}\rangle \mid \langle\mathcal{J}_7\rangle\,S \mid \langle\mathcal{J}_9\rangle\,S \mid \langle\mathcal{J}_{10}\rangle\,S \mid \langle\mathcal{L}_2^{-S}\rangle\,(\,S\,S\,) \mid S\,\langle\mathcal{L}_0^{-S}\rangle\,\langle\mathcal{N}^{-S}\rangle\\
\langle\mathcal{J}_1\rangle &::= A\\
\langle\mathcal{J}_2\rangle &::= \langle\mathcal{K}_0\rangle\,S \mid S\,\langle\mathcal{H}_0^{-S}\rangle \mid S\,S\,\langle\mathcal{J}_2\rangle\\
\langle\mathcal{J}_3\rangle &::= \langle\mathcal{K}_0\rangle\,S \mid S\,\langle\mathcal{L}_4^{-S}\rangle \mid S\,S\,\langle\mathcal{J}_3\rangle\\
\langle\mathcal{J}_4\rangle &::= S\,S\,(\,S\,S\,) \mid A\,S\\
\langle\mathcal{J}_5\rangle &::= S\,B\,(\,S\,S\,) \mid S\,S\,\langle\mathcal{J}_5\rangle\\
\langle\mathcal{J}_6\rangle &::= B\,(\,S\,S\,) \mid S\,S\,\langle\mathcal{J}_6\rangle
\end{aligned}
$$

$$\langle \mathcal{J}_7 \rangle ::= \langle \mathcal{K}_4 \rangle\, S \mid B\, \langle \mathcal{L}_0^{-S} \rangle \mid S\, S\, \langle \mathcal{J}_7 \rangle$$
$$\langle \mathcal{J}_8 \rangle ::= S\, \langle \mathcal{L}_0^{-S} \rangle \mid S\, S\, \langle \mathcal{J}_8 \rangle$$
$$\langle \mathcal{J}_9 \rangle ::= \langle \mathcal{J}_8 \rangle\, S \mid S\, S\, \langle \mathcal{J}_9 \rangle$$
$$\langle \mathcal{J}_{10} \rangle ::= S\, S\, \langle \mathcal{J}_{10} \rangle \mid S\, \langle \mathcal{L}_0^{-S} \rangle\, \langle \mathcal{L}_0^{-S} \rangle \mid SS\, (SS)\, \langle \mathcal{L}_0^{-S} \rangle \mid A\, S\, \langle \mathcal{L}_0^{-S} \rangle \mid B\, (SS)\, (SS) \mid \langle \mathcal{J}_{11} \rangle\, S$$
$$\langle \mathcal{J}_{11} \rangle ::= S\, S\, \langle \mathcal{J}_{11} \rangle \mid S\, \langle \mathcal{K}_4 \rangle \mid S\, A\, \langle \mathcal{L}_0^{-S} \rangle \mid S\, S\, (S\, S)\, A \mid A\, S\, A \mid \langle \mathcal{J}_{12} \rangle\, S$$
$$\langle \mathcal{J}_{12} \rangle ::= S\, S\, \langle \mathcal{J}_{12} \rangle \mid S\, S\, (S\, A) \mid S\, A\, (S\, S)$$

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Appendix

A Easy Facts

Proposition R. *For sets of S-terms $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{C}$, $(S\mathcal{A})^*[\mathcal{D}]\,\mathcal{C} \xrightarrow{*} (\mathcal{AC})^*[\mathcal{DC}]$. In particular, $(S\mathcal{A})^*[\mathcal{D}]\,\mathcal{C} \xrightarrow{\circledast} \mathcal{DC}$ so, if $\mathcal{DC}\!\uparrow$, $(S\mathcal{A})^*[\mathcal{D}]\,\mathcal{C}\!\uparrow$.*

Proof. By induction. *Inductive Base:* $(S\mathcal{A})^0[\mathcal{D}]\,\mathcal{C} = \mathcal{DC} = (\mathcal{AC})^0[\mathcal{DC}]$.

Inductive Step: For $k \geq 0$, if $(S\mathcal{A})^k[\mathcal{D}]\,\mathcal{C} \xrightarrow{k} (\mathcal{AC})^k[\mathcal{DC}]$,

$$(S\mathcal{A})^{k+1}[\mathcal{D}]\,\mathcal{C} \longrightarrow (\mathcal{AC})\big((S\mathcal{A})^k[\mathcal{D}]\,\mathcal{C}\big) \xrightarrow{k} (\mathcal{AC})\big((\mathcal{AC})^k[\mathcal{DC}]\big) = (\mathcal{AC})^{k+1}[\mathcal{DC}]\,. \qquad \square$$

Now, we can prove some preliminary results on non-normalizing S-terms:

Claim 1. $\mathcal{EE}\!\uparrow$.

Proof. To prove that $\mathcal{EE}\!\uparrow$ we will show that $\mathcal{EE} \xrightarrow{\oplus} \mathcal{EE}$. Using Proposition R, $\mathcal{EE} \xrightarrow{*} (S\mathcal{E})^*[\mathcal{Q}_2\mathcal{Q}_1\mathcal{E}]$. Thus, it will suffice to show $q\mathcal{Q}_1\mathcal{E} \to \mathcal{EE}$, for any $q \in \mathcal{Q}_2$. We need to consider three cases:

(i) Suppose $q \notin \mathcal{N}$. Then, q has a redex, $q \to \mathcal{Q}_2$, and $q\mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_2\mathcal{Q}_1\mathcal{E} \subseteq \mathcal{EE}$.

(ii) Suppose $q \in S(\mathcal{N} \cap \mathcal{Q}_1)$. Then, $q\mathcal{Q}_1\mathcal{E} \in S\mathcal{Q}_1\mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_1\mathcal{E}(\mathcal{Q}_1\mathcal{E}) \subseteq \mathcal{EE}$ (from *Remark 3*).

(iii) Suppose $q \in S\mathcal{NN}$. Then, $q\mathcal{Q}_1\mathcal{E} \in S\mathcal{NN}\mathcal{Q}_1\mathcal{E} \to \mathcal{N}\mathcal{Q}_1(\mathcal{N}\mathcal{Q}_1)\mathcal{E} \subseteq \mathcal{Q}_2\mathcal{Q}_1\mathcal{E} \subseteq \mathcal{EE}$. $\square$

Claim 2. $\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1\!\uparrow$.

Proof. To prove that $\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1\!\uparrow$ we will show that $\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1 \xrightarrow{\oplus} \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1 + \mathcal{EE}$. For $q \in \mathcal{Q}_3$ we need to consider the following three cases:

(i) Suppose $q \notin \mathcal{N}$. Then, q has a redex, $q \to \mathcal{Q}_3$, and $a\mathcal{Q}_2\mathcal{Q}_1 \to \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$.

(ii) Suppose $q \in S(\mathcal{N} \cap \mathcal{Q}_2)$. Then, $q\mathcal{Q}_2\mathcal{Q}_1 \in S\mathcal{Q}_2\mathcal{Q}_2\mathcal{Q}_1 \to \mathcal{Q}_2\mathcal{Q}_1(\mathcal{Q}_2\mathcal{Q}_1) \subseteq \mathcal{EE}$.

(iii) Suppose $q \in S\mathcal{NN}$. Then, $q\mathcal{Q}_2\mathcal{Q}_1 \in S\mathcal{NN}\mathcal{Q}_2\mathcal{Q}_1 \to \mathcal{N}\mathcal{Q}_2(\mathcal{N}\mathcal{Q}_2)\mathcal{Q}_1 \subseteq \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$. $\square$

Thus, in this technical preliminary session we established the following:

Corollary 1. $(\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1 + \mathcal{EE})\!\uparrow$

Recall: $B = S(SS)$, $\mathcal{Q}_3 = \overline{S + SS + B}$, $\mathcal{Q}_2 = B + \mathcal{Q}_3$, $\mathcal{Q}_1 = SS + \mathcal{Q}_2$, and $\mathcal{E} = (SS)^*[\mathcal{Q}_2\mathcal{Q}_1]$.

B Classification

For the proof of the theorem of this paper, we can limit ourselves to S-terms of the form $\mathcal{NN}$. That is because for any S-term m_1m_2 we can apply the algorithm recursively to m_1 and m_2 and answer *no* if either recursive call returns *no*, otherwise use the normal forms returned by those calls. To do that, we proceed now to classify all S-terms in $\mathcal{N}$ into different classes $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{L}_0$, $\mathcal{L}_1$, ... Then, the theorem is achieved by dividing the result into proofs that cover all combinations of normal forms from these classes. We will discuss the result such pairs, e.g., for $(\mathcal{H}_0, \mathcal{L}_0)$ whether $\mathcal{H}_0\mathcal{L}_0\!\uparrow$, in the next section.

To dissect $\mathcal{N}$, we first define the following disjoint sets:

$$\mathcal{H}_0 \stackrel{\text{def}}{=} (SS+B)^* \,[S+S\mathcal{N}+SBS+SB(SS)] \qquad \text{and}$$
$$\mathcal{H}_1 \stackrel{\text{def}}{=} (SS+B)^* \,[\mathcal{Q}_3\mathcal{Q}_2+S\mathcal{Q}_3\mathcal{M}]\,.$$

Some easy facts are $\mathcal{H}_0 \subseteq \mathcal{N}$, unlike $\mathcal{H}_1$, and $\mathcal{H}_0$, $\mathcal{H}_1$ are disjoint.

Claim 3. *$\mathcal{H}_0$ and $\mathcal{H}_1$ cover $\mathcal{N}$.*

Proof. We will show $\overline{\mathcal{H}_1} \cap \mathcal{N} \subseteq \mathcal{H}_0$ (in fact, $\overline{\mathcal{H}_1} \cap \mathcal{N} = \mathcal{H}_0$).

First note that if $x \in \mathcal{H}_1$, $SSx \in \mathcal{H}_1$ and $Bx \in \mathcal{H}_1$. Contrapositively, if $SSx \in \overline{\mathcal{H}_1}$ or $Bx \in \overline{\mathcal{H}_1}$, $x \in \overline{\mathcal{H}_1}$ must follow. Later, we will recall the previous statement as our initial observation.

Now, we expand:

$$\overline{\mathcal{Q}_3\mathcal{Q}_2} = S+S\mathcal{M}+SS\mathcal{M}+B\mathcal{M}+\mathcal{M}S+\mathcal{M}(SS) \qquad \text{and}$$
$$\overline{S\mathcal{Q}_3\mathcal{M}} = S+S\mathcal{M}+SS\mathcal{M}+B\mathcal{M}+SB\mathcal{M}+\mathcal{Q}_1\mathcal{M}\mathcal{M}\,.$$

That way, we can compute:

$$\overline{\mathcal{H}_1} \cap \mathcal{N} \subseteq (\,\overline{\mathcal{Q}_3\mathcal{Q}_2+S\mathcal{Q}_3\mathcal{M}}\,) \cap \mathcal{N} = S+S\mathcal{N}+SS\mathcal{N}+B\mathcal{N}+SBS+SB(SS)\,,$$

since $\mathcal{Q}_1\mathcal{M}\mathcal{M} \subseteq \overline{\mathcal{N}}$. Then, for any $x \in \overline{\mathcal{H}_1} \cap \mathcal{N}$ one of the following cases must apply:

$$\text{(i)}\quad x \in S+S\mathcal{N}+SBS+SB(SS) \qquad \text{or}$$
$$\text{(ii)}\quad x \in SS\mathcal{N}+B\mathcal{N}\,.$$

Case (i) coincides with the base of $\mathcal{H}_0$. Suppose case (ii), i.e., $x = SSx' \in \overline{\mathcal{H}_1}$ or $x = Bx' \in \overline{\mathcal{H}_1}$ for some $x' \in \mathcal{N}$. Then, from our initial observation, $x' \in \overline{\mathcal{H}_1}$. This pops an equivalent choice for x'. But case (ii) may apply recursively for only a finite number of times because each time the size of the term is reduced. Eventually, case (i) must be attained. Therefore:

$$x \in (SS+B)^*[S+S\mathcal{N}+SBS+SB(SS)] = \mathcal{H}_0\,. \qquad \square$$

We further refine and dissect $\mathcal{H}_1$ into more mutually disjoint sets:

$$\mathcal{L}_0 \stackrel{\text{def}}{=} (SS)^* \,[S+S\mathcal{N}]\,,$$
$$\mathcal{L}_1 \stackrel{\text{def}}{=} (SS)^* \,[BS+SBS]\,,$$
$$\mathcal{L}_2 \stackrel{\text{def}}{=} (SS)^* \,[B(SS)+BB]\,, \qquad \text{and}$$
$$\mathcal{L}_3 \stackrel{\text{def}}{=} (SS)^* \,[SB(SS)+B\mathcal{Q}_3]\,.$$

Some easy facts are $\mathcal{L}_{012} \stackrel{\text{def}}{=} \mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2 \subseteq \mathcal{H}_0$ and $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are mutually disjoint.

Claim 4. *$\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ cover $\mathcal{H}_0$.*

Proof. We will show that: $\overline{\mathcal{L}}_3 \cap \mathcal{H}_0 \subseteq \mathcal{L}_{012}$ —in fact $\overline{\mathcal{L}}_3 \cap \mathcal{H}_0 = \mathcal{L}_{012}$.

First, note that if $x \in \mathcal{L}_3$, $SSx \in \mathcal{L}_3$. Contrapositively, if $SSx \in \overline{\mathcal{L}}_3$, $x \in \overline{\mathcal{L}}_3$ must follow. Simultaneously, from the definition of $\mathcal{H}_0$ (and *Remark 2*), if $SSx \in \mathcal{H}_0$, $x \in \mathcal{H}_0$. So, if

$SSx \in \overline{\mathcal{L}}_3 \cap \mathcal{H}_0$, $x \in \overline{\mathcal{L}}_3 \cap \mathcal{H}_0$ must follow. Later, we will recall the previous statement as our initial observation.

Now, we expand:

$$\overline{B\mathcal{Q}_3} \;=\; S + S\mathcal{M} + SS\mathcal{M} + \mathcal{Q}_3\mathcal{M} + BS + B(SS) + BB\,.$$

That result combined with:

$$\mathcal{Q}_3\mathcal{M} \cap \mathcal{H}_0 \;=\; SBS + SB(SS)$$

facilitate the computation of:

$$\overline{\mathcal{L}}_3 \cap \mathcal{H}_0 \;\subseteq\; (\,\overline{SB(SS) + B\mathcal{Q}_3}\,) \cap \mathcal{H}_0 \;=\; S + S\mathcal{N} + SS\mathcal{N} + SBS + BS + B(SS) + BB\,.$$

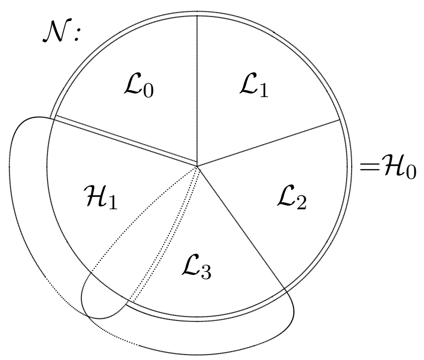
Then, for any $x \in \overline{\mathcal{L}}_3 \cap \mathcal{H}_0$ one of the following cases must apply:

$$\begin{aligned} &\text{(i)} \quad x \;\in\; S + S\mathcal{N} + BS + SBS + B(SS) + BB \qquad \text{or} \\ &\text{(ii)} \quad x \;\in\; SS\mathcal{N}\,. \end{aligned}$$

Case (i) coincides with the base of $\mathcal{L}_{012}$. Suppose case (ii), i.e., $x = SSx' \in \overline{\mathcal{L}}_3 \cap \mathcal{H}_0$ for some $x' \in \mathcal{N}$. Then, from our initial observation, $x' \in \overline{\mathcal{L}}_3 \cap \mathcal{H}_0$. This pops an equivalent choice for x'. But case (ii) may apply recursively for only a finite number of times because each time the size of the term is reduced. Eventually, case (i) must be attained. Therefore:

$$x \;\in\; (SS)^*[S + S\mathcal{N} + BS + SBS + B(SS) + BB\,] \;=\; \mathcal{L}_{012}\,. \qquad \square$$

From Claim 3 and Claim 4, we partition $\mathcal{N}$ according to the following diagram (the circle represents $\mathcal{N}$, double lines surround $\mathcal{H}_0$; notice that $\mathcal{H}_1$ and $\mathcal{L}_3$ intersect both inside and outside of $\mathcal{N}$, but that will not be a problem):



C The Proof

With these definitions and letting $\mathcal{L}_{23} \stackrel{\text{def}}{=} \mathcal{L}_2 + \mathcal{L}_3$ and $\mathcal{L}_{123} \stackrel{\text{def}}{=} \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ we will prove Theorem 1 by proving:

Part 1. $\mathcal{L}_0\,\mathcal{N}\!\downarrow$

Part 2. $\mathcal{H}_0\,\mathcal{L}_0\!\downarrow$

Part 3. $\mathcal{L}_1\,\mathcal{H}_0\!\downarrow$

Part 4. $\mathcal{L}_2\,\mathcal{L}_1\!\downarrow$

Part 5. $\mathcal{L}_{23}\,\mathcal{L}_{23}\!\uparrow$

Part 6. $\mathcal{L}_3\,\mathcal{L}_1\!\uparrow$

Part 7. $\mathcal{H}_1\,\mathcal{Q}_2\!\uparrow$

Part 8. $\mathcal{L}_{123}\,\mathcal{H}_1\!\uparrow$

Part 9. *There is a method to reduce any term in $\mathcal{N}(S{+}SS)$ to a form covered in one of Parts 1 through 8. In particular, for any $x \in (\mathcal{H}_1 \cap \mathcal{N})(S{+}SS)$ we can decide if x has a normal form.*

Proof of Part 1. Using Proposition R (and *Remark 1*):

$$\mathcal{L}_0\mathcal{N} \;=\; (SS)^*[\,S + S\mathcal{N}\,]\,\mathcal{N} \;\stackrel{*}{\longrightarrow}\; (S\mathcal{N})^*[\,S\mathcal{N} + S\mathcal{N}\mathcal{N}\,] \;\subseteq\; (S\mathcal{N})^*[\,\mathcal{N}\,] \;\subseteq\; \mathcal{N}\,. \qquad \square$$

Proof of Part 2. We show $\mathcal{H}_0\mathcal{L}_0\!\downarrow$ by induction on the structure of $\mathcal{H}_0$ and by using Part 1 (i.e., $\mathcal{L}_0\mathcal{N}\!\downarrow$). We will abbreviate the base of H_0: $\Theta = S + S\mathcal{N} + SBS + SB(SS)$.

Inductive Base: We divide $\Theta\mathcal{L}_0$ in the following cases:

$$\text{(i)} \qquad S\mathcal{L}_0 \subseteq S\mathcal{N} \subseteq \mathcal{N}\,.$$

$$\text{(ii)} \qquad S\mathcal{N}\mathcal{L}_0 \subseteq S\mathcal{N}\mathcal{N} \subseteq \mathcal{N}\,.$$

$$\begin{aligned}
\text{(iii)} \quad SBS\mathcal{L}_0 &\longrightarrow B\mathcal{L}_0(S\mathcal{L}_0) \\
&\stackrel{2}{\longrightarrow} S(\mathcal{L}_0(S\mathcal{L}_0))(S\mathcal{L}_0(\mathcal{L}_0(S\mathcal{L}_0))) \\
[\text{ Using Part 1, } \mathcal{L}_0(S\mathcal{L}_0) \stackrel{*}{\longrightarrow} \mathcal{N}\ ] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{N}(S\mathcal{L}_0\mathcal{N}) \subseteq \mathcal{N}\,.
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad SB(SS)\mathcal{L}_0 &\longrightarrow B\mathcal{L}_0(SS\mathcal{L}_0) \\
[\text{ Similar to (iii) }] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{N}(SS\mathcal{L}_0\mathcal{N}) \longrightarrow S\mathcal{N}(S\mathcal{N}(\mathcal{L}_0\mathcal{N})) \\
[\text{ Using Part 1, } \mathcal{L}_0\mathcal{N} \stackrel{*}{\longrightarrow} \mathcal{N}\ ] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{N}(S\mathcal{N}\mathcal{N})) \subseteq \mathcal{N}\,.
\end{aligned}$$

Inductive Step: For $k \geq 0$, we assume $(SS + B)^k[\Theta]\,\mathcal{L}_0 \stackrel{*}{\longrightarrow} \mathcal{N}$. Then:

$$\begin{aligned}
\text{(v)} \quad SS\big((SS{+}B)^k[\Theta]\,\big)\,\mathcal{L}_0 &\longrightarrow S\mathcal{L}_0\big((SS{+}B)^k[\Theta]\,\mathcal{L}_0\big) \\
[\text{ By inductive hypothesis }] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{L}_0\mathcal{N} \subseteq \mathcal{N}\,.
\end{aligned}$$

$$\begin{aligned}
\text{(vi)} \quad B\big((SS{+}B)^k[\Theta]\,\big)\,\mathcal{L}_0 &\stackrel{2}{\longrightarrow} S\big((SS{+}B)^k[\Theta]\,\mathcal{L}_0\big)\big(\mathcal{L}_0\big((SS{+}B)^k[\Theta]\,\mathcal{L}_0\big)\big) \\
[\text{ By inductive hypothesis }] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{N}(\mathcal{L}_0\mathcal{N}) \\
[\text{ Using Part 1 }] \qquad &\stackrel{*}{\longrightarrow} S\mathcal{N}\mathcal{N} \subseteq \mathcal{N}\,. \qquad \square
\end{aligned}$$

Proof of Part 3. We show $\mathcal{L}_1\mathcal{H}_0\!\downarrow$ using Part 2 (i.e., $\mathcal{H}_0\mathcal{L}_0\!\downarrow$). We start proving the claim for the base of $\mathcal{L}_1$ slit in two parts:

$$\begin{aligned} &\text{(i)} \quad BS\,\mathcal{H}_0 \xrightarrow{2} S(S\mathcal{H}_0)(\mathcal{H}_0(S\mathcal{H}_0)) \\ \text{[using PART 2, } \mathcal{H}_0(S\mathcal{H}_0) \xrightarrow{*} \mathcal{N} \text{]} \quad &\xrightarrow{*} S(S\mathcal{H}_0)\mathcal{N} \subseteq \mathcal{N}\,. \\ &\text{(ii)} \; SBS\,\mathcal{H}_0 \longrightarrow B\mathcal{H}_0(S\mathcal{H}_0) \\ &\xrightarrow{2} S(\mathcal{H}_0(S\mathcal{H}_0))(S\mathcal{H}_0(\mathcal{H}_0(S\mathcal{H}_0))) \\ &\xrightarrow{*} S\mathcal{N}(S\mathcal{H}_0\mathcal{N}) \subseteq \mathcal{N}\,. \end{aligned}$$

Then, using Proposition R,

$$\mathcal{L}_1\mathcal{H}_0 \;=\; (SS)^*[BS + SBS]\,\mathcal{H}_0 \;\xrightarrow{*}\; (S\mathcal{H}_0)^*[BS\mathcal{H}_0 + SBS\mathcal{H}_0] \;\xrightarrow{*}\; (S\mathcal{N})^*[\mathcal{N}] \;\subseteq\; \mathcal{N}\,. \qquad \square$$

Proof of Part 4. We show $\mathcal{L}_2\mathcal{L}_1\!\downarrow$ using Part 3 (i.e., $\mathcal{L}_1\mathcal{H}_0\!\downarrow$). We first prove the claim for the base of $\mathcal{L}_2$ slit in two parts:

$$\begin{aligned} &\text{(i)} \; B(SS)\,\mathcal{L}_1 \xrightarrow{2} S(SS\mathcal{L}_1)(\mathcal{L}_1(SS\mathcal{L}_1)) \\ \text{[using PART 3, } \mathcal{L}_1(SS\mathcal{L}_1) \xrightarrow{*} \mathcal{N} \text{]} \quad &\xrightarrow{*} S(SS\mathcal{L}_1)\mathcal{N} \subseteq \mathcal{N}\,. \\ &\text{(ii)} \quad BB\,\mathcal{L}_1 \xrightarrow{2} S(B\mathcal{L}_1)(\mathcal{L}_1(B\mathcal{L}_1)) \\ \text{[using PART 3, } \mathcal{L}_1(B\mathcal{L}_1) \xrightarrow{*} \mathcal{N} \text{]} \quad &\xrightarrow{*} S(B\mathcal{L}_1)\mathcal{N} \subseteq \mathcal{N}\,. \end{aligned}$$

Now, using Proposition R,

$$\mathcal{L}_2\mathcal{L}_1 \;=\; (SS)^*[B(SS) + BB]\,\mathcal{L}_1 \;\xrightarrow{*}\; (S\mathcal{L}_1)^*[B(SS)\mathcal{L}_1 + BB\mathcal{L}_1] \;\xrightarrow{*}\; (S\mathcal{N})^*[\mathcal{N}] \;\subseteq\; \mathcal{N}\,. \qquad \square$$

Proof of Part 5. The claim $\mathcal{L}_{23}\mathcal{L}_{23}\!\uparrow$ follows immediately from Claim 1 after proving: $\mathcal{L}_{23} = \mathcal{L}_2 + \mathcal{L}_3 \subseteq \mathcal{E}$. But this follows directly by combining the statements:

$$\text{(i)} \quad B(SS) + BB \subseteq \mathcal{Q}_2\mathcal{Q}_1 \qquad \text{and}$$

$$\text{(ii)} \quad SB(SS) + B\mathcal{Q}_3 \subseteq \mathcal{Q}_2\mathcal{Q}_1\,.$$

with the definition of $\mathcal{L}_2$ and $\mathcal{L}_3$ respectively. $\square$

Proof of Part 6. We will prove $\mathcal{L}_3\mathcal{L}_1\!\uparrow$ by letting $\mathcal{G} = \mathcal{Q}_3\mathcal{Q}_2$ and showing: $\mathcal{L}_3\mathcal{L}_1 \xrightarrow{\oplus} \mathcal{L}_1\mathcal{G} \xrightarrow{\oplus} \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$ the result follows from that statement and Claim 2. First, to examine the base of $\mathcal{L}_3$, we check:

$$\text{(i)} \quad SB(SS)\mathcal{L}_1 \longrightarrow B\mathcal{L}_1(SS\mathcal{L}_1) \subseteq B\mathcal{Q}_3\mathcal{L}_1 \qquad \text{and}$$

$$\text{(ii)} \quad B\mathcal{Q}_3\mathcal{L}_1 \xrightarrow{2} S(\mathcal{Q}_3\mathcal{L}_1)(\mathcal{L}_1(\mathcal{Q}_3\mathcal{L}_1)))\,.$$

Then, by Proposition R:

$$\mathcal{L}_3\mathcal{L}_1 \xrightarrow{\circledast} (SB(SS) + B\mathcal{Q}_3)\mathcal{L}_1 \xrightarrow{\oplus} \mathcal{L}_1(\mathcal{Q}_3\mathcal{L}_1) \subseteq \mathcal{L}_1\mathcal{G}\,.$$

Now we split the base of $\mathcal{L}_1$ in the two cases:

$$\text{(iii)} \quad BS\mathcal{G} \xrightarrow{2} S(S\mathcal{G})(\mathcal{G}(S\mathcal{G})) \qquad \text{and}$$

$$\text{(iv)} \quad SBS\mathcal{G} \longrightarrow B\mathcal{G}(S\mathcal{G}) \xrightarrow{2} S(\mathcal{G}(S\mathcal{G}))(S\mathcal{G}(\mathcal{G}(S\mathcal{G})))\,.$$

Then, by Proposition R:

$$\mathcal{L}_1\mathcal{G} \xrightarrow{\circledast} (BS + SBS)\mathcal{G} \xrightarrow{\oplus} \mathcal{G}(S\mathcal{G}) \subseteq \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1\,. \qquad \square$$

Proof of Part 7. To prove $\mathcal{H}_1\mathcal{Q}_2\!\uparrow$ we notice that: $S\mathcal{Q}_3\mathcal{M}\mathcal{Q}_2 \rightarrow \mathcal{Q}_3\mathcal{Q}_2(\mathcal{M}\mathcal{Q}_2) \subseteq \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$. Then, by Proposition R,

$$\mathcal{H}_1\mathcal{Q}_2 \xrightarrow{\circledast} (\mathcal{Q}_3\mathcal{Q}_2 + S\mathcal{Q}_3\mathcal{M})\,\mathcal{Q}_2 \xrightarrow{*} \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1\,.$$

This result combined with Claim 2 completes the proof. □

Proof of Part 8. To prove $\mathcal{L}_{123}\mathcal{H}_1\!\uparrow$ we first note that:

$$\mathcal{L}_{123} \subseteq (SS)^*(B\mathcal{M} + SB\mathcal{M})\,.$$

So combining the cases:

$$\text{(i)}\quad B\mathcal{M}\mathcal{H}_1 \xrightarrow{2} S(\mathcal{M}\mathcal{H}_1)(\mathcal{H}_1(\mathcal{M}\mathcal{H}_1)) \qquad \text{and}$$

$$\text{(ii)}\quad SB\mathcal{M}\mathcal{H}_1 \longrightarrow B\mathcal{H}_1(\mathcal{M}\mathcal{H}_1) \xrightarrow{2} S(\mathcal{H}_1(\mathcal{M}\mathcal{H}_1))((\mathcal{M}\mathcal{H}_1)(\mathcal{H}_1(\mathcal{M}\mathcal{H}_1)))$$

with Proposition R we get:

$$\mathcal{L}_{123}\mathcal{H}_1 \xrightarrow{\circledast} (B\mathcal{M} + SB\mathcal{M})\mathcal{H}_1 \xrightarrow{\oplus} \mathcal{H}_1(\mathcal{M}\mathcal{H}_1) \subseteq \mathcal{H}_1\mathcal{Q}_2\,.$$

This fact combined with Part 7 completes this proof. □

Proof of Part 9. First, we will prove that for any S-term $n \in \mathcal{N}$ we can decide if $n(SS)$ has a normal form using induction on the size of n. As inductive base, if $|n| = 1$, $n = S$ so $n(SS) = S(SS)$ is in normal form. For the inductive step, we suppose that the statement is true for all S-terms of length $\leq k$. Then, if $|n| = k + 1$ we need to check two cases:

(i) Suppose $n = Sn_1$. Then, $n(SS) = Sn_1(SS)$ is in normal form.

(ii) Suppose $n = Sn_2n_3$. Then, $n(SS) = Sn_2n_3(SS) \rightarrow n_2(SS)(n_3(SS))$. But by inductive hypothesis, we can decide if $n_2(SS) \xrightarrow{*} n_4 \in \mathcal{N}$ and $n_3(SS) \xrightarrow{*} n_5 \in \mathcal{N}$. If that is the case, $n(SS) \xrightarrow{*} n_4n_5$ and we can decide if n_4n_5 has a normal form by one of Parts 1-8 (note that $n_5 \neq SS$ because $|n_5| \geq |n_3(SS)|$). Otherwise, n does not have normal form.

A similar inductive proof shows that for any $n \in \mathcal{N}$ we can decide if nS has a normal form. As inductive base, if $|n| = 1$, $n = S$ so $nS = SS$ is in normal form. For the inductive step, we suppose that the statement is true for all S-terms of length $\leq k$. Then, if $|n| = k + 1$ we need to check two cases:

(i) Suppose $n = Sn_1$. Then, $nS = Sn_1S$ is in normal form.

(ii) Suppose $n = Sn_2n_3$. Then, $nS = Sn_2n_3S \rightarrow n_2S(n_3S)$. But by inductive hypothesis we can decide if $n_2S \xrightarrow{*} n_4 \in \mathcal{N}$ and $n_3S \xrightarrow{*} n_5 \in \mathcal{N}$. If that is the case, $nS \xrightarrow{*} n_4n_5$ and we can decide if n_4n_5 has a normal form by one of Parts 1-8 or the first part of this proof (note that $n_5 \neq S$ because $|n_5| \geq |n_3S|$). Otherwise, n does not have normal form. □

D A Grammar for $\langle\mathcal{H}_0\rangle$

From this point on, we use the angle brackets $\langle\cdot\rangle$ to denote the set of *"predecessors"* for a given set. That is, for any set $\mathcal{A}$,

$$\langle\mathcal{A}\rangle \stackrel{\text{def}}{=} \{\, x \in \mathcal{M} \mid x \xrightarrow{*} \mathcal{A} \,\}\,.$$

It is our objective to develop a context free grammar to recognize $\langle\mathcal{N}\rangle$. Naturally, we will use these sets of predecessors as non-terminal symbols in our grammar.

The terms in $\langle\mathcal{N}\rangle$ are either S or terms of from $\langle\mathcal{N}\rangle\langle\mathcal{N}\rangle$. To describe the terms of the second form we use the classification of $\mathcal{N}$ into $\mathcal{H}_0$ and $\mathcal{H}_1$, and the further classification of $\mathcal{H}_0$ into $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$. With these classes, the applications described in PARTS 1 through 9 cover all possible terms in $\mathcal{N}\mathcal{N}$. Because of the incomplete nature of the result in PART 9 we need to the define the following sets:

$$\mathcal{N}^{-S} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid nS \stackrel{*}{\longrightarrow} \mathcal{N}\,\} \qquad \text{and}$$
$$\mathcal{N}^{-SS} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid n(SS) \stackrel{*}{\longrightarrow} \mathcal{N}\,\}\,.$$

Then, the results from PARTS 1-9 prove the sufficiency of:

$$\langle\mathcal{N}\rangle \ ::= \ S \mid \langle\mathcal{L}_0\rangle\,\langle\mathcal{N}\rangle \mid \langle\mathcal{H}_0\rangle\,\langle\mathcal{L}_0\rangle \mid \langle\mathcal{L}_1\rangle\,\langle\mathcal{H}_0\rangle \mid \langle\mathcal{L}_2\rangle\,\langle\mathcal{L}_1\rangle \mid \langle\mathcal{N}^{-S}\rangle\,S \mid \langle\mathcal{N}^{-SS}\rangle\,(\,S\,S\,)$$

(Here, the production symbol ::= can be substituted by equality when the disjunction symbols | are substituted by union.) In this section we will expand $\langle\mathcal{H}_0\rangle$.

Before proceeding, we need:

CLAIM 5. *The sets $\mathcal{Q}_3\mathcal{Q}_2$, $S\mathcal{Q}_3\mathcal{M}$, and, by extension, $\mathcal{H}_1$ are closed under reduction, that is, if a term in either set reduces, the resulting term lies in the same set.*

Proof. If SSm or Bm has a redex, it must be a sub-term of m, otherwise $SSm + Bm \subseteq S\mathcal{N}\mathcal{N}$, i.e., each term would be in normal form already. So, we only need to show the result for $\mathcal{Q}_3\mathcal{Q}_2$ and $S\mathcal{Q}_3\mathcal{M}$.

If $m = Sm_1m_2 \in S\mathcal{Q}_3\mathcal{M}$ has a redex, such redex must be a sub-term of m_1, with $m_1 \to m_1'$, or a sub-term of m_2, with $m_2 \to m_2'$. Either way, $m' = Sm_1'm_2$ or $m' = Sm_1m_2'$ satisfies $m' \in S\mathcal{Q}_3\mathcal{M}$ because if $m_1 \to m_1'$, $m_1' \in \mathcal{Q}_3$ —since S, SS, or B are no redecta.

If $m = m_1m_2 \in \mathcal{Q}_3\mathcal{Q}_2$ has a redex, such redex must be a sub-term of m_1, with $m_1 \to m_1'$, a sub-term of m_2, with $m_2 \to m_2'$, or the redex is the entire term $m_1m_2 = Sm_3m_4\,m_2$, with $m_1 = Sm_3m_4$. The cases $m' = m_1'm_2$ and $m' = m_1m_2'$ satisfy $m' \in \mathcal{Q}_3\mathcal{Q}_2$ because in either case $m_1' \in \mathcal{Q}_3$ or $m_2' \in \mathcal{Q}_3 \subseteq \mathcal{Q}_3$ because no redex reduces into S, SS, or B. Finally $m' = m_3m_2(m_4m_2)$ also satisfies $m' \in \mathcal{Q}_3\mathcal{Q}_2$ because both m_3m_2 and $m_4m_2 \in \mathcal{Q}_3$ because $m_2 \in \mathcal{Q}_2$. □

The result above reveals that $\langle\mathcal{H}_0\rangle$ and $\mathcal{Q}_3\mathcal{Q}_2$ are disjoint. Since $\mathcal{H}_0 \subseteq \mathcal{N}$, this fact is equivalent to $\langle\mathcal{H}_0\rangle \subseteq (S{+}SS{+}B)\langle\mathcal{N}\rangle + \langle\mathcal{N}\rangle(S{+}SS)$. This will facilitate the grammatical description of $\langle\mathcal{H}_0\rangle$. For this, we introduce the following sets:

$$\mathcal{H}_0^{-S} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid nS \stackrel{*}{\longrightarrow} \mathcal{H}_0\,\} \qquad \text{and}$$
$$\mathcal{H}_0^{-SS} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid n(SS) \stackrel{*}{\longrightarrow} \mathcal{H}_0\,\}\,.$$

Hence, we can describe $\langle\mathcal{H}_0\rangle$ completely with:

$$\langle\mathcal{H}_0\rangle \ ::= \ S \mid S\,\langle\mathcal{N}\rangle \mid S\,S\,\langle\mathcal{H}_0\rangle \mid B\,\langle\mathcal{H}_0\rangle \mid \langle\mathcal{H}_0^{-S}\rangle\,S \mid \langle\mathcal{H}_0^{-SS}\rangle\,(\,S\,S\,)$$

because the application of other pairs of strings in $\langle\mathcal{N}\rangle\langle\mathcal{N}\rangle$ results in $\mathcal{H}_1$. However, we are left with the task of producing rules for $\langle\mathcal{H}_0^{-S}\rangle$ and $\langle\mathcal{H}_0^{-SS}\rangle$. We start with:

Claim 6.

$$\mathcal{H}_0^{-SS} = (SS)^*[S + SS + B + SB] .$$

Proof. Easily, Proposition R verifies:

$$(SS)^*[S + SS + B + SB](SS) \xrightarrow{*} (B)^*[B + SS(SS) + B(SS) + SB(SS)] \subseteq \mathcal{H}_0 .$$

Suppose that $n(SS) \xrightarrow{*} \mathcal{H}_0$ for some $n \in \mathcal{N} = S + S\mathcal{N} + S\mathcal{N}\mathcal{N}$. For any $n \in S\mathcal{Q}_3$ we have $n(SS) \in S\mathcal{Q}_3\mathcal{M} \subseteq \mathcal{H}_1$, which is a contradiction. However, all $n \in (S + S\mathcal{N}) - S\mathcal{Q}_3 = S + SS + B + SB$ satisfy $n(SS) \in B + SS(SS) + B(SS) + SB(SS) \subseteq \mathcal{H}_0$, so $n \in \mathcal{H}_0^{-SS}$. This choice of values coincides with the base for the terms in $\mathcal{H}_0^{-SS}$ we want.

If $n \in S\mathcal{N}\mathcal{N}$, $n = Sn_1n_2$ for some $n_1, n_2 \in \mathcal{N}$. Then, $n(SS) \to n_1(SS)(n_2(SS))$. This way, $n_1 = S$, or else $n_1(SS)(n_2(SS)) \in \mathcal{Q}_3\mathcal{Q}_2 \subseteq \mathcal{H}_1$, which is a contradiction. With $n = SSn_2$, $n(SS) \to B(n_2(SS))$ but this still requires $n_2(SS) \xrightarrow{*} \mathcal{H}_0$. This means any term $n \in \mathcal{H}_0^{-SS}$ may have SS as prefix any number of times, but its base must be some $n' \in (S + S\mathcal{N}) \cap \mathcal{H}_0^{-SS}$. □

Readily, $\mathcal{H}_0^{-SS} \subseteq \mathcal{L}_0 \subseteq \mathcal{H}_0$. We classify the terms in $\mathcal{H}_0^{-SS}$ as follows: $S \in \mathcal{H}_0^{-SS}$; the terms in $\mathcal{H}_0^{-SS}$ of the form $S\mathcal{N}$ are $SS + B + SB$; and the terms in $\mathcal{H}_0^{-SS}$ of the form $S\mathcal{N}\mathcal{N}$ are the terms in $SS\mathcal{H}_0^{-SS}$.

Claim 7.

$$\{ n \in \mathcal{N} \mid n(SS) \xrightarrow{*} \mathcal{H}_0^{-SS} \} = S + SS .$$

Proof. Clearly, $(S + SS)(SS) = B + SS(SS) \xrightarrow{*} \mathcal{H}_0^{-SS}$. Now, suppose $n(SS) \xrightarrow{*} n_1$ for some $n \in \mathcal{N}$ with $n_1 \in \mathcal{H}_0^{-SS} = (SS)^*[S + SS + B + SB]$. Then, $n_1 \in \mathcal{H}_0$, because $\mathcal{H}_0^{-SS} \subseteq \mathcal{H}_0$, so $n(SS) \xrightarrow{*} \mathcal{H}_0$ and $n \in \mathcal{H}_0^{-SS}$. Then, n_1 is within of normal forms reduced from $\mathcal{H}_0^{-SS}(SS)$, i.e., $n_1 \in (B)^*[B + SS(SS) + B(SS) + SB(SS)]$ —using Proposition R. By comparing the expressions that describe the sets where n_1 belongs, we deduce $n_1 \in B + SS(SS)$, therefore $n \in S + SS$. □

Let

$$\mathcal{K}_0 \stackrel{\text{def}}{=} (SS)^*[S] , \qquad \mathcal{K}_1 \stackrel{\text{def}}{=} (SS)^*[SS] , \quad \text{and} \qquad \mathcal{K}_{01} \stackrel{\text{def}}{=} \mathcal{K}_0 + \mathcal{K}_1 = (SS)^*[S + SS] .$$

Claim 8.

$$\{ n \in \mathcal{N} \mid nS \xrightarrow{*} \mathcal{H}_0^{-SS} \} = \mathcal{K}_{01} .$$

Proof. Let $\mathcal{G} = \{ n \in \mathcal{N} \mid nS \xrightarrow{*} \mathcal{H}_0^{-SS} \}$. Suppose $nS \xrightarrow{*} \mathcal{H}_0^{-SS}$ for some $n \in \mathcal{N} = S + S\mathcal{N} + S\mathcal{N}\mathcal{N}$.

If $n \in S\mathcal{N}\mathcal{N}$, $n = Sn_1n_2$ for some $n_1, n_2 \in \mathcal{N}$. Then, $nS \to n_1S(n_2S)$. This way, $n_1 = S$ or $n_2 = S$, otherwise $n_1S(n_2S) \in \mathcal{Q}_3\mathcal{Q}_2 \subseteq \mathcal{H}_1$, which is a contradiction. If $n_1 = S$, $n = SSn_2$ and $nS \to SS(n_2S)$. Recalling that the only terms in $\mathcal{H}_0^{-SS}$ of the form $S\mathcal{N}\mathcal{N}$ are in $SS\mathcal{H}_0^{-SS}$ we find that we still require $n_2S \xrightarrow{*} \mathcal{H}_0^{-SS}$. By this means we conclude that any term $n \in \mathcal{G}$ may have SS as prefix any number of times, but its base must be some $n' \in (S + S\mathcal{N} + S\mathcal{N}S) \cap \mathcal{G}$.

We check the base for the terms in $\mathcal{G}$ in three steps. First, we check $S \in \mathcal{G}$ by recalling $SS \in \mathcal{H}_0^{-SS}$. Second, if $n \in S\mathcal{N}$, $n = Sn_3$, for some $n_3 \in \mathcal{N}$, and $nS = Sn_3S \in S\mathcal{N}\mathcal{N}$. However the only terms in $\mathcal{H}_0^{-SS}$ of the form $S\mathcal{N}\mathcal{N}$ are in $SS\mathcal{H}_0^{-SS}$. Thus $nS = SSS$ and $n = SS$. Third, if $n \in S\mathcal{N}S$, $n = Sn_1S$, for some $n_1 \in \mathcal{N}$, and $nS \longrightarrow n_1S(SS) \xrightarrow{*} \mathcal{H}_0^{-SS}$. Hence, because of Claim 7, $n_1S \in S + SS$, $n_1 = S$, and $n = SSS$. Summarizing, $(S + S\mathcal{N} + S\mathcal{N}S) \cap \mathcal{G} = S + SS + SSS$. This checks that the base for the terms in $\mathcal{G}$ is as we want. □

Claim 9.

$$\mathcal{H}_0^{-S} = (SS)^*[S + SS + B + SB + S\,\mathcal{K}_{01}\,S]\,.$$

Proof. Using Claim 8 we show: $S\mathcal{K}_{01}SS \longrightarrow \mathcal{K}_{01}S(SS) \xrightarrow{*} \mathcal{H}_0^{-SS}(SS) \xrightarrow{*} \mathcal{H}_0$. Then, Proposition R verifies:

$$(SS)^*[S + SS + B + SB + S\mathcal{K}_{01}S]\,S \xrightarrow{*}$$
$$(SS)^*[SS + SSS + BS + SBS + S\mathcal{K}_{01}SS] \xrightarrow{*} (SS)^*[\mathcal{H}_0] \subseteq \mathcal{H}_0\,.$$

Suppose that $nS \xrightarrow{*} \mathcal{H}_0$ for some $n \in \mathcal{N} = S + S\mathcal{N} + S\mathcal{N}\mathcal{N}$. If $n \in S\mathcal{N}\mathcal{N}$, $n = Sn_1n_2$ for some n_1, $n_2 \in \mathcal{N}$. Then, $nS \to n_1S(n_2S)$. This way $n_1 = S$ or $n_2 = S$, otherwise $n_1S(n_2S) \in \mathcal{Q}_3\mathcal{Q}_2 \subseteq \mathcal{H}_1$, which is a contradiction. If $n_1 = S$, $n = SSn_2$ and $nS \to SS(n_2S)$, but we still require $n_2S \xrightarrow{*} \mathcal{H}_0$. This means any term $n \in \mathcal{H}_0^{-S}$ may have SS as prefix any number of times, but its base must be some $n' \in (S + S\mathcal{N} + S\mathcal{N}S) \cap \mathcal{H}_0^{-S}$.

If $n = Sn_1S \in S\mathcal{N}S$, $nS \to n_1S(SS) \in \mathcal{H}_0$. Thus $n_1S \in \mathcal{H}_0^{-SS}$, $n_1 \in \mathcal{K}_{01}$, and $n \in S\mathcal{K}_{01}S$. For any $n \in S\mathcal{Q}_3$ we have $nS \in S\mathcal{Q}_3\mathcal{M} \subseteq \mathcal{H}_1$, which is a contradiction. However, for all $n \in (S + S\mathcal{N}) - S\mathcal{Q}_3 = S + SS + B + SB$ we have $nS \in SS + SSS + BS + SBS \subseteq \mathcal{H}_0$. This checks that the base for the terms in $\mathcal{H}_0^{-S}$ is as we want. □

We can define $\mathcal{L}_0^{-S}$, $\mathcal{L}_1^{-S}$, $\mathcal{L}_2^{-S}$, and $\mathcal{L}_3^{-S}$ in a manner similar to that of $\mathcal{H}_0^{-S}$. I.e.,

$$\mathcal{L}_i^{-S} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid nS \xrightarrow{*} \mathcal{L}_i \,\}$$

for $i = 0, 1, 2, 3$. With these definitions:

Corollary 2.

$$\mathcal{L}_0^{-S} = (SS)^*[S + SS] = \mathcal{K}_{01}\,,$$
$$\mathcal{L}_1^{-S} = (SS)^*[B + SB]\,,$$
$$\mathcal{L}_2^{-S} = (SS)^*[BS] \subseteq (SS)^*[S\,\mathcal{K}_{01}\,S]\,, \qquad \textit{and}$$
$$\mathcal{L}_3^{-S} = (SS)^*[S\,\mathcal{K}_{01}\,S] - (SS)^*[SSS] - (SS)^*[BS]\,.$$

Proof. This partition of $\mathcal{H}_0^{-S}$ may be verified by direct computation and using Proposition R. □

For any S-term m if $m = m_lm_r$ for some terms m_l and m_r, we say m_r is a *right sub-term* of m. We extend the notion of right sub-term to include its reflexive transitive closure. We now show some results about right sub-term ahead.

Claim 10. *Given any S-term m. For any m' such that $m \xrightarrow{*} m'$, every $n \in \mathcal{N}$ that is a right sub-term of m is also a right sub-term of m'.*

Proof. Without loss of generality, we can assume $m = m_1m_2 \in \mathcal{M}$ for some m_1, m_2 and $m \to m'$. With this assumption, the right sub-term n must be a right sub-term of m_2. In this reduction, the redex must be a sub-term of m_1, with $m_1 \to m_1'$, a sub-term of m_2, with $m_2 \to m_2'$, or the redex is the entire term, $m_1m_2 = Sm_3m_4\,m_2$, with $m_1 = Sm_3m_4$. In the first case $m' = m_1'm_2$ and n is readily a right sub-term of $m_1'm_2$. In the second case $m' = m_1m_2'$, we assume that we can recursively prove the claim for m_2. Then, n is a right-sub-term of m_2' and thus it is a right sub-term of m_1m_2'. In the third case $m' = m_3m_2(m_4m_2)$, clearly n is a right sub-term of m_4m_2 and also of m'. □

Corollary 3. *For any $n \in \mathcal{N}$, suppose $mc \xrightarrow{*} n$ for some $m \in \mathcal{M}$ and $c \in \mathcal{N}$. Then, c is a right sub-term of n.*

Corollary 4. *Suppose m is the redectum reduced from some redex $Sadc$ with $c \in \mathcal{N}$ and $m \xrightarrow{*} m_1 m_2$. Then c is a proper right sub-term of both m_1 and m_2.*

Remark 4. Given $\mathcal{A} = (SS)^*[\mathcal{D}]$ with $\mathcal{D} \subseteq \mathcal{N}$, suppose $xy \xrightarrow{*} \mathcal{A}$ for some x and y. Then, either $xy \xrightarrow{*} \mathcal{D}$, or $x = SS$, or $y = S$. This is justified as follows: The first two options are trivial. If none of those two options are satisfied, we would face the reduction $xy \xrightarrow{*} SSa \in \mathcal{A}$ (for some $a \in \mathcal{A}$!). In such case, Corollary 4 imposes the third choice by stating (the normal form of) y is a proper right sub-term of SS. (Note for any $d \in \mathcal{D}$, we can search for all pairs (x, y) so that $xy \xrightarrow{*} d$ by exhausting all pairs that satisfy $|xy| \leq |d|$.)

Corollary 4 shows that no term is $S\mathcal{N}S$ is a redectum. Neither terms in $S + S\mathcal{N}$ are redecta. After this, *Remark 4* above establishes how to determine all xy such that $xy \xrightarrow{*} (SS)^*[S + S\mathcal{N} + B + SB + S\mathcal{L}_0^{-S}S] = \mathcal{H}_0^{-S}$ (recall Claim 9 and Corollary 2). For instance, let:

$$\mathcal{K}_2 \stackrel{\text{def}}{=} (SS)^*[S\,\mathcal{L}_0^{-S}] \qquad \text{and} \qquad \mathcal{K}_3 \stackrel{\text{def}}{=} (SS)^*[S\,\mathcal{L}_0^{-S}\,S]\,.$$

For these, Proposition R tells us:

$$\mathcal{K}_2\,S \xrightarrow{*} \mathcal{K}_3\,.$$

while *Remark 4* ensures no other reductions result in $\mathcal{K}_3$. That ensures the sufficiency of:

$$\langle \mathcal{K}_3 \rangle \ ::= \ \langle \mathcal{K}_2 \rangle\, S \mid S\,S\,\langle \mathcal{K}_3 \rangle$$

This way, we can produce a "complete" context free grammar for $\langle \mathcal{H}_0 \rangle$ using on the following set of rules:

$$
\begin{array}{rlcl}
\text{(i)} & \langle \mathcal{H}_0 \rangle & ::= & S \mid S\,\langle \mathcal{N} \rangle \mid S\,S\,\langle \mathcal{H}_0 \rangle \mid B\,\langle \mathcal{H}_0 \rangle \mid \langle \mathcal{H}_0^{-S} \rangle\, S \mid \langle \mathcal{H}_0^{-SS} \rangle\,(\,S\,S\,) \\
\text{(ii)} & \langle \mathcal{H}_0^{-S} \rangle & ::= & \langle \mathcal{H}_0^{-SS} \rangle \mid \langle \mathcal{K}_3 \rangle \\
\text{(iii)} & \langle \mathcal{H}_0^{-SS} \rangle & ::= & \langle \mathcal{L}_0^{-S} \rangle \mid \langle \mathcal{L}_1^{-S} \rangle \\
\text{(iv)} & \langle \mathcal{L}_0^{-S} \rangle & ::= & \langle \mathcal{K}_0 \rangle \mid \langle \mathcal{K}_1 \rangle \\
\text{(v)} & \langle \mathcal{L}_1^{-S} \rangle & ::= & B \mid S\,B \mid S\,S\,\langle \mathcal{L}_1^{-S} \rangle \\
\text{(vi)} & \langle \mathcal{K}_3 \rangle & ::= & \langle \mathcal{K}_2 \rangle\, S \mid S\,S\,\langle \mathcal{K}_3 \rangle \\
\text{(vii)} & \langle \mathcal{K}_2 \rangle & ::= & \langle \mathcal{K}_1 \rangle \mid S\,\langle \mathcal{L}_0^{-S} \rangle \mid S\,S\,\langle \mathcal{K}_2 \rangle \\
\text{(viii)} & \langle \mathcal{K}_1 \rangle & ::= & \langle \mathcal{K}_0 \rangle\, S \mid S\,S\,\langle \mathcal{K}_1 \rangle \\
\text{(ix)} & \langle \mathcal{K}_0 \rangle & ::= & S \mid \langle \mathcal{K}_1 \rangle\, S \mid S\,S\,\langle \mathcal{K}_0 \rangle \\
\text{(x)} & B & ::= & S\,(\,S\,S\,)
\end{array}
$$

Our grammar for $\langle \mathcal{N} \rangle$ also needs rules for $\langle \mathcal{L}_0 \rangle$, $\langle \mathcal{L}_1 \rangle$, and $\langle \mathcal{L}_2 \rangle$. These rules also follow easily after *Remark 4*. We have shown already $\mathcal{L}_0^{-S}$, $\mathcal{L}_1^{-S}$, and $\mathcal{L}_2^{-S}$ in Corollary 2. Indeed, we already have rules for $\langle \mathcal{L}_0^{-S} \rangle$ and $\langle \mathcal{L}_1^{-S} \rangle$. However, $\mathcal{L}_2^{-S}$ is simply presented as a subset of $\mathcal{K}_3$. For that reason, we introduce:

$$\mathcal{K}_4 \stackrel{\text{def}}{=} (SS)^*[B]\,.$$

Then (note $SSS(SS) \to BB \in \mathcal{L}_2$ but no $xy \to B(SS)\,!$),

$$
\begin{array}{rrcl}
\text{(xi)} & \langle\mathcal{L}_0\rangle & ::= & S \mid S\,\langle\mathcal{N}\rangle \mid \langle\mathcal{L}_0^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_0\rangle \\
\text{(xii)} & \langle\mathcal{L}_1\rangle & ::= & \langle\mathcal{L}_1^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_1\rangle \\
\text{(xiii)} & \langle\mathcal{L}_2\rangle & ::= & B\,(\,S\,S\,) \mid S\,S\,S\,(\,S\,S\,) \mid B\,B \mid \langle\mathcal{L}_2^{-S}\rangle\,S \mid S\,S\,\langle\mathcal{L}_2\rangle \\
\text{(xiv)} & \langle\mathcal{L}_2^{-S}\rangle & ::= & \langle\mathcal{K}_4\rangle\,S \mid S\,S\,\langle\mathcal{L}_2^{-S}\rangle \\
\text{(xv)} & \langle\mathcal{K}_4\rangle & ::= & B \mid S\,S\,\langle\mathcal{K}_4\rangle
\end{array}
$$

E Beyond $\mathcal{H}_0$: The sets $\mathcal{N}^{-S}$ and $\mathcal{N}^{-SS}$

Recall:

$$
\mathcal{N}^{-S} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid nS \stackrel{*}{\longrightarrow} \mathcal{N} \,\} \qquad \text{and}
$$

$$
\mathcal{N}^{-SS} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid n(SS) \stackrel{*}{\longrightarrow} \mathcal{N} \,\}\,.
$$

Since $S+SS \subseteq \mathcal{L}_0$, Part 2, $\mathcal{H}_0\,\mathcal{L}_0\!\downarrow$, easily shows $\mathcal{H}_0 \subseteq \mathcal{N}^{-S}$ and $\mathcal{H}_0 \subseteq \mathcal{N}^{-SS}$. Now, we proceed to complete the representation of the terms in $\mathcal{N}^{-SS}$. Let:

$$
\mathcal{L}_4 \stackrel{\text{def}}{=} (SS+B)^*[S\,\mathcal{H}_0^{-SS}(S+SS)]\,.
$$

We will show $\mathcal{N}^{-SS} = \mathcal{H}_0 + \mathcal{L}_4$. After this, we can revise the grammatical rule for $\langle\mathcal{N}\rangle$ to:

$$
\langle\mathcal{N}\rangle \;::=\; S \mid \langle\mathcal{L}_0\rangle\,\langle\mathcal{N}\rangle \mid \langle\mathcal{H}_0\rangle\,\langle\mathcal{L}_0\rangle \mid \langle\mathcal{L}_1\rangle\,\langle\mathcal{H}_0\rangle \mid \langle\mathcal{L}_2\rangle\,\langle\mathcal{L}_1\rangle \mid \langle\mathcal{N}^{-S}\rangle\,S \mid \langle\mathcal{L}_4\rangle\,(\,S\,S\,)
$$

Indeed, we show:

Part 10.

$$
\mathcal{L}_4(SS)\!\downarrow \qquad \textit{and} \qquad (\,\mathcal{H}_1 \cap \mathcal{N} - \mathcal{L}_4)\,(SS)\!\uparrow\,.
$$

Proof. Recalling Part 2, $\mathcal{H}_0\,\mathcal{L}_0\!\downarrow$,

$$
S\,\mathcal{H}_0^{-SS}(S+SS)\,(SS) \;\longrightarrow\; (\,\mathcal{H}_0^{-SS}(SS))\,(S(SS)+SS(SS)) \;\stackrel{*}{\longrightarrow}\; \mathcal{H}_0\,\mathcal{L}_0 \;\stackrel{*}{\longrightarrow}\; \mathcal{N}\,.
$$

Then, the base of $\mathcal{L}_4$, $S\,\mathcal{H}_0^{-SS}(S+SS) \subseteq \mathcal{N}^{-SS}$. To extend this to the rest of $\mathcal{L}_4$,

$$
B\,\mathcal{N}^{-SS}(SS) \;\stackrel{2}{\longrightarrow}\; S\,(\,\mathcal{N}^{-SS}(SS))\,(SS\,(\,\mathcal{N}^{-SS}(SS))) \;\stackrel{*}{\longrightarrow}\; S\,\mathcal{N}\,(SS\mathcal{N}) \subseteq \mathcal{N} \qquad \text{and}
$$

$$
SS\,\mathcal{N}^{-SS}(SS) \;\longrightarrow\; B\,(\,\mathcal{N}^{-SS}(SS)) \;\stackrel{*}{\longrightarrow}\; B\,\mathcal{N} \subseteq \mathcal{N}
$$

prove $(SS+B)\mathcal{N}^{-SS} \subseteq \mathcal{N}^{-SS}$, which generalizes to $(SS+B)^*[\mathcal{N}^{-SS}] \subseteq \mathcal{N}^{-SS}$. This expands our initial result for the base of $\mathcal{L}_4$ to $\mathcal{L}_4 \subseteq \mathcal{N}^{-SS}$ and equivalently $\mathcal{L}_4(SS)\!\downarrow$.

Now, we have to show $(\,\mathcal{H}_1 \cap \mathcal{N} - \mathcal{L}_4)\,(SS)\!\uparrow$. Because of Proposition R, we only need to show $((S\mathcal{Q}_3\mathcal{M} + \mathcal{Q}_3\mathcal{Q}_2) \cap \mathcal{N} - \mathcal{L}_4)\,(SS)\!\uparrow$. Clearly, $\mathcal{Q}_3\mathcal{Q}_2 \cap \mathcal{N} - \mathcal{L}_4 \subseteq \mathcal{Q}_3\mathcal{Q}_2$. Then, $(\mathcal{Q}_3\mathcal{Q}_2 \cap \mathcal{N} - \mathcal{L}_4)(SS)\!\uparrow$ because of Claim 2, $\mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1\!\uparrow$, and $SS \in \mathcal{Q}_1$. We have left to show $(S\mathcal{Q}_3\mathcal{M} \cap \mathcal{N} - \mathcal{L}_4)\,(SS)\!\uparrow$. For that purpose, we show that $n \in S\mathcal{Q}_3\mathcal{M} \cap \mathcal{N}^{-SS}$ only if $n \in \mathcal{L}_4$.

Let $n \in S\mathcal{Q}_3\mathcal{M} \cap \mathcal{N}^{-SS}$. We can write $n = Sn_1n_2$ for some $n_1 \in \mathcal{Q}_3 \cap \mathcal{N}$ and $n_2 \in \mathcal{N}$. Suppose $n_2 \in \overline{S+SS}$. Then, $n = Sn_1n_2 \in \mathcal{Q}_3\mathcal{Q}_2$ and if so, $n(SS)\!\uparrow$, because of Claim 2.

Hence, $n_2 \in S+SS$, so $n \in Sn_1(S+SS)$. To satisfy $n_1 \in \mathcal{Q}_3 \cap \mathcal{N}$ we have two choices: $n_1 = Sn_3$ for some $n_3 \in \mathcal{Q}_2 \cap \mathcal{N}$, or $n_1 = Sn_4n_5$ for some $n_4, n_5 \in \mathcal{N}$. In the first case, $n_1 = Sn_3$ for some $n_3 \in \mathcal{Q}_2 \cap \mathcal{N}$, we have:

$$n(SS) \in S(Sn_3)(S+SS)(SS) \longrightarrow S\, n_3\, (SS)\, ((S+SS)(SS)) \longrightarrow n_3\, ((S+SS)(SS))\, (SS((S+SS)(SS)))\,.$$

If $n_3 \in \mathcal{Q}_3$ the reduction above would show $n(SS) \xrightarrow{2} \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$ and so $n(SS)\!\uparrow$, which is a contradiction. Therefore $n_3 \in \mathcal{Q}_2 - \mathcal{Q}_3 = B$, so

$$n \in S(SB)(S+SS) \subseteq S\,\mathcal{H}_0^{-SS}(S+SS) \subseteq \mathcal{L}_4\,.$$

In the second case, with $n_1 = Sn_4n_5$ for some $n_4, n_5 \in \mathcal{N}$,

$$n(SS) \in S(Sn_4n_5)(S+SS)(SS) \longrightarrow S\, n_4\, n_5\, (SS)\, ((S+SS)(SS)) \longrightarrow n_4\, (SS)\, (n_5(SS))\, ((S+SS)(SS))\,.$$

If $n_4 \in \mathcal{Q}_1$ the reduction above would show $n(SS) \xrightarrow{2} \mathcal{Q}_3\mathcal{Q}_2\mathcal{Q}_1$ and so $n(SS)\!\uparrow$, which is a contradiction. Thus $n_4 \in \mathcal{N} - \mathcal{Q}_1 = S$, so

$$n(SS) \xrightarrow{2} B\,(n_5(SS))\,((S+SS)(SS))$$

Let $n_5' \in \mathcal{N}$ be such that $n_5(SS) \xrightarrow{*} n_5'$. Either $n_5' \in \mathcal{H}_0$ or $n_5' \in \mathcal{H}_1 \cap \mathcal{N}$. If $n' \in \mathcal{H}_1$ the reduction above would show $n(SS) \xrightarrow{+} \mathcal{H}_1\mathcal{Q}_2$ and so, because of Part 7, $n(SS)\!\uparrow$, which is a contradiction. Therefore $n_5' \in \mathcal{H}_0$ and equivalently $n_5 \in \mathcal{H}_0^{-SS}$, so

$$n \in S(SS\,\mathcal{H}_0^{-SS})(S+SS) \subseteq S\,\mathcal{H}_0^{-SS}(S+SS) \subseteq \mathcal{L}_4\,. \qquad \square$$

Corollary 5.

$$\mathcal{H}_1 \cap \mathcal{N}^{-SS} = (SS+B)^*[S(SB+SS\,\mathcal{H}_0^{-SS})(S+SS)]\,.$$

The right sub-terms of $\mathcal{L}_4$ are S, SS, and $\mathcal{L}_4$. Because of Corollary 3, for the grammar of $\langle\mathcal{L}_4\rangle$ we only need to describe the sets:

$$\mathcal{L}_4^{-S} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid nS \xrightarrow{*} \mathcal{L}_4 \,\} \qquad \text{and}$$
$$\mathcal{L}_4^{-SS} \stackrel{\text{def}}{=} \{\, n \in \mathcal{N} \mid n(SS) \xrightarrow{*} \mathcal{L}_4 \,\}$$

so we could write:

$$\langle\mathcal{L}_4\rangle \;::=\; S\,S\,\langle\mathcal{L}_4\rangle \mid B\,\langle\mathcal{L}_4\rangle \mid \langle\mathcal{L}_4^{-S}\rangle\, S \mid \langle\mathcal{L}_4^{-SS}\rangle\,(\,S\,S\,)$$

We start by proving:

Claim 11.

$$\mathcal{L}_4^{-SS} = (SS)^*[S\,\mathcal{H}_0^{-SS}]\,.$$

Proof. Easily, PROPOSITION R verifies:

$$(SS)^*[S\,\mathcal{H}_0^{-SS}]\,(SS) \;\xrightarrow{*}\; (B)^*[S\,\mathcal{H}_0^{-SS}(SS)] \;\subseteq\; \mathcal{L}_4\,.$$

For the complement, assume that $n(SS) \xrightarrow{*} \mathcal{H}_0$ for some $n \in \mathcal{N} = S + S\mathcal{N} + S\mathcal{N}\mathcal{N}$. Clearly $n \neq S$ because $S(SS) \notin \mathcal{L}_4$. Suppose $n \in S\mathcal{N}$. Then, $n(SS) \in S\mathcal{N}(SS)$ so $n(SS) \in S\,\mathcal{H}_0^{-SS}(SS)$ and $n \in S\,\mathcal{H}_0^{-SS}$. Suppose $n \in S\mathcal{N}\mathcal{N}$. Then, $n = Sn_1n_2$ for some n_1, $n_2 \in \mathcal{N}$ and $n(SS) \to n_1(SS)(n_2(SS))$. For this, $n_1 = S$, because otherwise, $n_1(SS)(n_2(SS)) \in \mathcal{Q}_3\mathcal{Q}_2$ but, $\mathcal{Q}_3\mathcal{Q}_2$ being closed under reduction (CLAIM 5) and $\mathcal{L}_4$ being a subset of $(SS{+}B)\mathcal{N} + \mathcal{N}(S{+}SS)$ make that a contradiction. With $n = SSn_2$, $n(SS) \to B(n_2(SS))$ but this still requires $n_2(SS) \xrightarrow{*} \mathcal{L}_4$. This means any term $n \in \mathcal{L}_4^{-SS}$ may have SS as prefix any number of times but eventually its base must be some $n' \in (S + S\mathcal{N}) \cap \mathcal{L}_4^{-SS}$. □

Recall:

$$\mathcal{K}_0 \;\overset{\text{def}}{=}\; (SS)^*[S] \;\subseteq\; \mathcal{K}_{01} \;\overset{\text{def}}{=}\; (SS)^*[S + SS] \;=\; \mathcal{L}_0^{-S}\,.$$

CLAIM 12.

$$\{\, n \in \mathcal{N} \mid nS \in \mathcal{L}_4^{-SS} \,\} \;=\; \mathcal{K}_0\,.$$

Proof. Easily, PROPOSITION R verifies:

$$\mathcal{K}_0\,S \;=\; (SS)^*[S]\,S \;\xrightarrow{*}\; (SS)^*[SS] \;\subseteq\; (SS)^*[S\,\mathcal{H}_0^{-SS}] \;=\; \mathcal{L}_4^{-SS}\,.$$

For the complement, note that $\mathcal{L}_4^{-SS} \subseteq \mathcal{L}_0$ so $\{\, n \in \mathcal{N} \mid nS \in \mathcal{L}_4^{-SS} \,\} \subseteq \mathcal{L}_0^{-S} = (SS)^*[S + SS]$. But $(SS)^*[SS]\,S \xrightarrow{*} (SS)^*[S]$, which is disjoint from $\mathcal{L}_4^{-SS}$. □

CLAIM 13.

$$\mathcal{L}_4^{-S} \;=\; (SS)^*[S\,\mathcal{H}_0^{-SS} \;+\; S\mathcal{K}_0S\,]\,.$$

Proof. Using CLAIM 12 we show: $S\mathcal{K}_0SS \longrightarrow \mathcal{K}_0S(SS) \xrightarrow{*} \mathcal{L}_4^{-SS}(SS) \xrightarrow{*} \mathcal{L}_4$. Easily, PROPOSITION R verifies:

$$(SS)^*[S\,\mathcal{H}_0^{-SS} + S\mathcal{K}_0S\,]\,S \;\xrightarrow{*}\; (SS)^*[S\,\mathcal{H}_0^{-SS}S + S\mathcal{K}_0SS\,] \;\xrightarrow{*}\; (SS)^*[\mathcal{L}_4] \;\subseteq\; \mathcal{L}_4\,.$$

For the complement, assume $nS \xrightarrow{*} \mathcal{H}_0$ for some $n \in \mathcal{N} = S + S\mathcal{N} + S\mathcal{N}\mathcal{N}$. Clearly, $n \neq S$, because $SS \notin \mathcal{L}_4$. Suppose $n \in S\mathcal{N}$. Then, $nS \in S\mathcal{N}S$ so $nS \in S\,\mathcal{H}_0^{-SS}S$ and $n \in S\,\mathcal{H}_0^{-SS}$. Suppose $n \in S\mathcal{N}\mathcal{N}$. Then, $n = Sn_1n_2$ for some n_1, $n_2 \in \mathcal{N}$ and $nS \to n_1S(n_2S)$. This needs $n_1 = S$ or $n_2 = S$, for otherwise $n_1S(n_2S) \in \mathcal{Q}_3\mathcal{Q}_2$ but, $\mathcal{Q}_3\mathcal{Q}_2$ being closed under reduction (CLAIM 5) and $\mathcal{L}_4$ being a subset of $(SS{+}B)\mathcal{N}{+}\mathcal{N}(S{+}SS)$ make that a contradiction. Suppose $n_2 = S$. Then, $nS = Sn_1SS \to n_1S(SS)$ so $n_1S \in \mathcal{L}_4^{-SS}$ and $n_1 \in \mathcal{K}_0$, i.e., $n \in S\mathcal{K}_0S$. Suppose $n_2 \neq S$. Then, $n_1 = S$, $nS = SSn_2S \to SS(n_2S)$ but this still requires $n_2S \xrightarrow{*} \mathcal{L}_4$. This means any term $n \in \mathcal{L}_4^{-S}$ may have SS as prefix any number of times but eventually its base must be some $n' \in (S + S\mathcal{N} + S\mathcal{N}S) \cap \mathcal{L}_4^{-S}$. □

Note $\mathcal{L}_4^{-SS} \subseteq \mathcal{L}_4^{-S}$ and the difference $\mathcal{L}_4^{-S} - \mathcal{L}_4^{-SS} = (SS)^*[S\mathcal{K}_0S] \subseteq \mathcal{H}_0^{-S}$ so $\{\, n \in \mathcal{N} \mid nS \xrightarrow{*} \mathcal{H}_1 \cap \mathcal{N}^{-SS} \} \;\subseteq\; \mathcal{L}_4^{-SS}$. Let:

$$\mathcal{K}_5 \;\overset{\text{def}}{=}\; (SS)^*[S\,\mathcal{K}_0] \qquad \text{and} \qquad \mathcal{K}_6 \;\overset{\text{def}}{=}\; (SS)^*[S\,\mathcal{K}_0\,S]\,.$$

Then, we can specify a grammar for $\langle \mathcal{L}_4 \rangle$:

$$\begin{array}{rlcl}
\text{(xvi)} & \langle \mathcal{L}_4 \rangle & ::= & S\,S\,\langle \mathcal{L}_4 \rangle \mid B\,\langle \mathcal{L}_4 \rangle \mid \langle \mathcal{L}_4^{-S} \rangle\,S \mid \langle \mathcal{L}_4^{-SS} \rangle\,(\,S\,S\,) \\
\text{(xvii)} & \langle \mathcal{L}_4^{-S} \rangle & ::= & \langle \mathcal{L}_4^{-SS} \rangle \mid \langle \mathcal{K}_6 \rangle \\
\text{(xviii)} & \langle \mathcal{L}_4^{-SS} \rangle & ::= & \langle \mathcal{K}_0 \rangle\,S \mid S\,\langle \mathcal{H}_0^{-SS} \rangle \mid S\,S\,\langle \mathcal{L}_4^{-SS} \rangle \\
\text{(xix)} & \langle \mathcal{K}_6 \rangle & ::= & \langle \mathcal{K}_5 \rangle\,S \mid S\,S\,\langle \mathcal{K}_6 \rangle \\
\text{(xx)} & \langle \mathcal{K}_5 \rangle & ::= & \langle \mathcal{K}_0 \rangle\,S \mid S\,\langle \mathcal{K}_0 \rangle \mid S\,S\,\langle \mathcal{K}_5 \rangle
\end{array}$$

Now, the only piece missing in our grammar for $\langle \mathcal{N} \rangle$ is $\langle \mathcal{N}^{-S} \rangle$. To fill this gap, we first prove:

Part 11.

$$\mathcal{N}^{-S} \;=\; (S\mathcal{L}_0^{-S})^*[S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,]\,.$$

Proof. Readily:

$$\begin{aligned}
&(S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S)\;S \;\xrightarrow{*} \\
&\quad SS + S\mathcal{N}S + (\mathcal{H}_0^{-S}S)(\mathcal{L}_0^{-S}S) + (\mathcal{L}_1^{-S}S)(\mathcal{H}_0^{-S}S) + (\mathcal{L}_2^{-S}S)(\mathcal{L}_1^{-S}S) + (\mathcal{L}_4^{-S}S)(SS) \;\xrightarrow{*} \\
&\qquad SS + S\mathcal{N}S + \mathcal{H}_0\mathcal{L}_0 + \mathcal{L}_1\mathcal{H}_0 + \mathcal{L}_2\mathcal{L}_1 + \mathcal{L}_4(SS) \;\xrightarrow{*}\; \mathcal{N}\,,
\end{aligned}$$

because of Parts 2, 3, 4 and 10. Thus,

$$S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S \;\subseteq\; \mathcal{N}^{-S}\,.$$

Now,

$$S\mathcal{L}_0^{-S}\mathcal{N}^{-S}S \;\longrightarrow\; (\mathcal{L}_0^{-S}\,S)(\mathcal{N}^{-S}\,S) \;\xrightarrow{*}\; \mathcal{L}_0\mathcal{N} \;\xrightarrow{*}\; \mathcal{N}$$

because of Part 1. Induction based on this gives us:

$$\big((S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}]\,\big)\,S \;\xrightarrow{*}\; \mathcal{N}\,.$$

Therefore,

$$(S\mathcal{L}_0^{-S})^*[S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,] \;\subseteq\; \mathcal{N}^{-S}\,.$$

Obviously,

$$S + S\mathcal{N} \;\subseteq\; (S\mathcal{L}_0^{-S})^*[S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,]\,.$$

We need to prove for $n \in \mathcal{N}^{-S} \cap S\mathcal{N}\mathcal{N}$,

$$n \;\in\; (S\mathcal{L}_0^{-S})^*[S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,]\,.$$

For $n \in \mathcal{N}^{-S}$ suppose $n = Sn_1n_2$ for some $n_1, n_2 \in \mathcal{N}$. Then, $nS \to (n_1S)(n_2S)$. Let $n_1', n_2' \in \mathcal{N}$ be such that $n_1S \xrightarrow{*} n_1'$ and $n_2S \xrightarrow{*} n_2'$, so $nS \xrightarrow{*} n_1'n_2'$. Now we recall the covering of $\mathcal{N}$ by $\mathcal{H}_0$ and $\mathcal{H}_1$, and the further covering of $\mathcal{H}_0$ by $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$. With this coverings we can safely state

$$n_1'n_2' \;\in\; \mathcal{L}_0\mathcal{N} + \mathcal{H}_0\mathcal{L}_0 + \mathcal{L}_1\mathcal{H}_0 + \mathcal{L}_2\mathcal{L}_1 + \mathcal{L}_{23}\mathcal{L}_{23} + \mathcal{L}_3\mathcal{L}_1 + \mathcal{H}_1\mathcal{Q}_2 + \mathcal{L}_{123}\mathcal{H}_1 + \mathcal{H}_1(S{+}SS)\,.$$

The terms at the right are the the expressions in PARTS 1-9. We cover the alternatives from these terms leaving $\mathcal{L}_0\mathcal{N}$ to be the last.

The alternatives $n_1'n_2' \in \mathcal{H}_0\mathcal{L}_0 + \mathcal{L}_1\mathcal{H}_0 + \mathcal{L}_2\mathcal{L}_1$ occur only if $n = Sn_1n_2 \in S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S}$. The alternatives $n_1'n_2' \in \mathcal{L}_{23}\mathcal{L}_{23} + \mathcal{L}_3\mathcal{L}_1 + \mathcal{H}_1\mathcal{Q}_2 + \mathcal{L}_{123}\mathcal{H}_1$ imply $n_1'n_2'\uparrow$ because of PARTS 5 through 8, but this is a contradiction, so we reject them. The alternative $n_1'n_2' \in \mathcal{H}_1(S{+}SS)$ forces $n_2' = SS$ and so, because of PART 10, $n_1' \in \mathcal{L}_4$. Therefore, $n_1 \in \mathcal{L}_4^{-S}$ and $n \in S\mathcal{L}_4^{-S}S$. This far, we have shown for $n \in \mathcal{N}^{-S} \cap S\mathcal{N}\mathcal{N}$,

$$n \;\in\; S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S \qquad \text{or} \qquad nS \;\xrightarrow{*}\; \mathcal{L}_0\mathcal{N}\,.$$

Finally, the alternative $n_1'n_2' \in \mathcal{L}_0\mathcal{N}$ needs $n_1 \in \mathcal{L}_0^{-S}$ and $n_2 \in \mathcal{N}^{-S}$, i.e., $n \in S\mathcal{L}_0^{-S}\mathcal{N}^{-S}$. This means that n may have any number of prefixes from $S\mathcal{L}_0^{-S}$ but ultimately its base must be some $n' \in \mathcal{N}^{-SS}$ covered by the previous alternatives or $n' \in S + S\mathcal{N}$. Therefore,

$$\mathcal{N}^{-S} \;\subseteq\; (S\mathcal{L}_0^{-S})^*[S + S\mathcal{N} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,]\,. \qquad \square$$

Remark 5. Recall $\mathcal{H}_0 \subseteq \mathcal{N}^{-S} \cap \mathcal{N}^{-SS}$. Now, we can easily check:

$$\mathcal{L}_4 \;=\; (SS+B)^*[S\mathcal{H}_0^{-SS}(S+SS)] \;\subseteq\; (S\mathcal{L}_0^{-S})^*[S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S}] \;\subseteq\; \mathcal{N}^{-S}\,.$$

It is no surprise $\mathcal{N}^{-SS} \subseteq \mathcal{N}^{-S}$.

After this result, to investigate $\langle\mathcal{N}^{-S}\rangle$ we only need investigate the redexes that reduce into $\mathcal{N}^{-S}$. Suppose n_1n_2 is a redex with $n_1, n_2 \in \mathcal{N}$, and $n_1n_2 \xrightarrow{+} n_0 \in \mathcal{N}^{-S}$. Since n_0 is a redectum, $n_0 \notin S + S\mathcal{N} + S\mathcal{N}S$. Then, from the expression for $\mathcal{N}^{-S}$ in PART 11, $n_0 \in S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S}$. In short, $n_1n_2 \xrightarrow{+} S\mathcal{H}_0^{-S}\mathcal{N}^{-S}$. We proceed to analyze exhaustively the choices for n_1: We have either $n_1 \in \mathcal{H}_0$, or $n_1 \in \mathcal{H}_1 \cap \mathcal{N}^{-SS}$ and $n_2 = SS$, or $n_1 \in \mathcal{H}_1 \cap \mathcal{N}^{-S}$ and $n_2 = S$.

#1. Suppose $n_1 \in \mathcal{H}_0$. We need to examine various cases from $n_1 \in (SS)^*[S + S\mathcal{N} + SBS + SB(SS) + B\mathcal{H}_0]$:

(i) Suppose $n_1 \in (SS)^*[S]$. If $n_1 = S$, the S-term $n_1n_2 \in S\mathcal{N}$ ($\subseteq \mathcal{N}^{-S}$!) is not a redex. Then, we are supposing $n_1 = (SS)^{k+1}[S] = SS\big((SS)^k[S]\big)$ for some $k \geq 0$. Then,

$$n_1n_2 \;=\; SS\big((SS)^k[S]\,\big)\,n_2 \;\longrightarrow\; Sn_2\big((SS)^k[S]\,n_2\big) \;\xrightarrow{*}\; S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} \;\subseteq\; \mathcal{N}^{-S}\,.$$

Therefore, either $n_2 \in \mathcal{L}_0^{-S}$ (and $(SS)^k[S]\,n_2 \xrightarrow{*} \mathcal{N}^{-S}$) or $(SS)^k[S]\,n_2 \xrightarrow{*} \mathcal{H}_0^{-S}$.

- Suppose $n_2 \in \mathcal{L}_0^{-S}$. Then, we may verify:

$$n_1\,n_2 \;\in\; (SS)^*[S]\;\mathcal{L}_0^{-S} \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[S\mathcal{L}_0^{-S}] \;\subseteq\; (S\mathcal{L}_0^{-S})^*[S\mathcal{N}\,] \;\subseteq\; \mathcal{N}^{-S}\,.$$

- Suppose $(SS)^k[S]\,n_2 \xrightarrow{*} \mathcal{H}_0^{-S}$ and $n_2 \notin \mathcal{L}_0^{-S}$. Recalling $\mathcal{H}_0^{-S} = (SS)^*[S{+}SS{+}B{+}SB{+}S\mathcal{L}_0^{-S}S\,]$ and the rules for $\langle\mathcal{H}_0^{-S}\rangle$, we determine this needs $(SS)^k[S] = S$ and $n_2 = B$. Given this, we verify:

$$n_1\,n_2 \;=\; (SSS)B \;\longrightarrow\; SB(SB) \;\subseteq\; S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} \;\subseteq\; \mathcal{N}^{-S}\,.$$

(ii) Suppose $n_1 \in (SS)^*[Sn'_1]$ for some $n'_1 \in \mathcal{N}$. If $n_1 = Sn'_1$, the S-term $n_1 n_2 \in S\mathcal{N}\mathcal{N}$ is not a redex (the allowed values for n'_1 and n_2 can be found in Part 11). Then, we are supposing $n_1 = (SS)^{k+1}[Sn'_1] = SS\big((SS)^k[Sn'_1]\big)$ for some $k \geq 0$. Then,

$$n_1\, n_2 \;=\; SS\big((SS)^k[Sn'_1]\big)\, n_2 \;\longrightarrow\; Sn_2\big((SS)^k[Sn'_1]\, n_2\big) \;\xrightarrow{*}\; S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} \;\subseteq\; \mathcal{N}^{-S}\,.$$

Therefore, either $n_2 \in \mathcal{L}_0^{-S}$ (and $(SS)^k[Sn_1]\, n_2 \xrightarrow{*} \mathcal{N}^{-S}$) or $(SS)^k[Sn_1]\, n_2 \xrightarrow{*} \mathcal{H}_0^{-S}$.

- Suppose $n_2 \in \mathcal{L}_0^{-S}$. Then,

$$n_1\, n_2 \in (SS)^*[Sn'_1]\; \mathcal{L}_0^{-S} \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[Sn'_1\mathcal{L}_0^{-S}] \;\subseteq\; (S\mathcal{L}_0^{-S})^*[Sn'_1\mathcal{L}_0^{-S}] \;\subseteq\; \mathcal{N}^{-S}\,.$$

  However, from Part 11, for the above statement, we need to match,

$$Sn'_1\mathcal{L}_0^{-S} \;\subseteq\; S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} + S\mathcal{L}_4^{-S}S\,.$$

  This is satisfied only if $n'_1 \in \mathcal{H}_0^{-S}$, for any $n_2 \in \mathcal{L}_0^{-S}$, or if $n'_1 \in \mathcal{L}_4^{-S}$, for the particular case of $n_2 = S \in \mathcal{L}_0^{-S}$. These alternatives are verified with:

$$n_1\, n_2 \;\in\; (SS)^*[S\mathcal{H}_0^{-S}]\; \mathcal{L}_0^{-S} \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S}] \;\subseteq\; \mathcal{N}^{-S} \qquad \text{and}$$

$$n_1\, n_2 \;\in\; (SS)^*[S\mathcal{L}_4^{-S}]\, S \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[S\mathcal{L}_4^{-S}S] \;\subseteq\; \mathcal{N}^{-S}\,.$$

- Suppose $(SS)^k[Sn'_1]\, n_2 \xrightarrow{*} \mathcal{H}_0^{-S}$ and $n_2 \notin \mathcal{L}_0^{-S}$. Then, $(SS)^k[Sn'_1] = SS$ and $n_2 \in \mathcal{H}_0^{-S} - \mathcal{L}_0^{-S}$. With $n_1 = SS(SS)$ and $n_2 \in \mathcal{L}_i^{-S}$ for $i = 1$, 2, or 3,

$$n_1\, n_2 \;\in\; SS(SS)\, \mathcal{L}_i^{-S} \;\longrightarrow\; S\mathcal{L}_i^{-S}(SS\mathcal{L}_i^{-S}) \;\subseteq\; S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} \;\subseteq\; \mathcal{N}^{-S}\,.$$

  This is only satisfied when $i = 1$ (note $i = 0$ is not an option now!). Then, we may verify,

$$n_1\, n_2 \;\in\; SS(SS)\, \mathcal{L}_1^{-S} \;\longrightarrow\; S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} \;\subseteq\; \mathcal{N}^{-S}\,.$$

(iii) Suppose $n_1 \in (SS)^*[SBS]$. Then,

$$n_1\, n_2 \;=\; (SS)^*[SBS]\, n_2 \;\xrightarrow{*}\; (Sn_2)^*[SBS\, n_2] \;\xrightarrow{*}\; \mathcal{N}^{-S}\,.$$

For this, we need to verify $SBS\, n_2 \xrightarrow{*} \mathcal{N}^{-S}$ first:

$$SBS\, n_2 \;\longrightarrow\; Bn_2(Sn_2) \;\xrightarrow{2}\; S\,\big(n_2(Sn_2)\big)\,\big(Sn_2(n_2(Sn_2))\big) \;\xrightarrow{*}\; S\mathcal{H}_0^{-S}\mathcal{N}^{-S}\,.$$

Thus, $n_2(Sn_2) \xrightarrow{*} \mathcal{H}_0^{-S}$. To satisfy this, we need $n_2 \in S + SS$. This being provided, we can verify $SBS\, n_2 \xrightarrow{*} \mathcal{N}^{-S}$ with:

$$SBSS \xrightarrow{*} SB(SSB) \in S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} \subseteq \mathcal{N}^{-S} \qquad \text{but}$$

$$SBS(SS) \notin \langle\mathcal{N}^{-S}\rangle, \text{ because } SBS(SS)S\!\uparrow \text{ (proof from part 8 after some reductions).}$$

Therefore, given $n_1 \in (SS)^*[SBS]$, only for $n_2 = S$ we may verify:

$$n_1\, n_2 \;\in\; (SS)^*[SBS]\, S \;\xrightarrow{*}\; (SS)^*[SBS\, S] \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}] \;\subseteq\; \mathcal{N}^{-S}\,.$$

(iv) Suppose $n_1 = (SS)^*[SB(SS)]$. Then:

$$n_1\, n_2 \;=\; (SS)^*[SB(SS)]\, n_2 \;\xrightarrow{*}\; (Sn_2)^*[SB(SS)\, n_2] \;\xrightarrow{*}\; \mathcal{N}^{-S}.$$

For this, we need to verify $SB(SS)\, n_2 \xrightarrow{*} \mathcal{N}^{-S}$ first:

$$\begin{aligned} SB(SS)\, n_2 \;\longrightarrow\; Bn_2(SSn_2) \;\xrightarrow{2}\; S\left(n_2(SSn_2)\right)\left(SSn_2(n_2(SSn_2))\right) \;\longrightarrow \\ S\left(n_2(SSn_2)\right)\left(S\left(n_2(SSn_2)\right)\left(n_2(n_2(SSn_2))\right)\right) \;\xrightarrow{*}\; S\mathcal{H}_0^{-S}\mathcal{N}^{-S}. \end{aligned}$$

Thus, $n_2(SSn_2) \xrightarrow{*} \mathcal{H}_0^{-S}$, but for this we need $n_2 = SS$. Then:

$$\begin{aligned} SB(SS)\, n_2 \xrightarrow{*} S\left(SS(SS(SS))\right)\left(S\left(SS(SS(SS))\right)\left(SS(SS(SS(SS)))\right)\right) \;\in \\ (S\mathcal{L}_0^{-S})^*[S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S}] \;\subseteq\; \mathcal{N}^{-S}. \end{aligned}$$

Therefore, given $n_1 \in (SS)^*[SB(SS)]$, only for $n_2 = SS$ we may verify:

$$\begin{aligned} n_1\, n_2 \;\in\; (SS)^*[SB(SS)]\,(SS) \;\xrightarrow{*} \\ (S(S{+}SS))^*[SB(SS)\,(SS)] \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}] \;\subseteq\; \mathcal{N}^{-S}. \end{aligned}$$

(v) Suppose $n_1 = (SS)^*Bn_1'$ for some $n_1' \in \mathcal{H}_0$. Then,

$$n_1\, n_2 \;=\; (SS)^*[Bn_1']\, n_2 \;\xrightarrow{*}\; (Sn_2)^*[Bn_1'n_2] \;\xrightarrow{*}\; \mathcal{N}^{-S}.$$

For this, we need to verify $Bn_1'n_2 \xrightarrow{*} \mathcal{N}^{-S}$ first:

$$Bn_1'n_2 \;\xrightarrow{2}\; S(n_1'n_2)(n_2(n_1'n_2)) \;\xrightarrow{*}\; S\mathcal{L}_0^{-S}\mathcal{N}^{-S} + S\mathcal{H}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{L}_1^{-S}\mathcal{H}_0^{-S} + S\mathcal{L}_2^{-S}\mathcal{L}_1^{-S} \;\subseteq\; \mathcal{N}^{-S}.$$

Therefore, either $n_1'n_2 \in \mathcal{L}_0^{-S}$ (and $n_2(n_1'n_2) \xrightarrow{*} \mathcal{N}^{-S}$) or $n_2(n_1'n_2) \xrightarrow{*} \mathcal{H}_0^{-S}$.

- Suppose $n_1'n_2 \in \mathcal{L}_0^{-S}$. Then, either $n_1' \in \mathcal{L}_0^{-S}$ and $n_2 = S$, or $n_1' = SS$ and $n_2 \in \mathcal{L}_0^{-S}$. For these alternatives we compute:

$$B\mathcal{L}_0^{-S}S \;\xrightarrow{2}\; S(\mathcal{L}_0^{-S}S)(S(\mathcal{L}_0^{-S}S)) \;\longrightarrow\; S\mathcal{L}_0^{-S}(S\mathcal{L}_0^{-S}) \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[S\mathcal{N}\,] \subseteq \mathcal{N}^{-S} \quad \text{and}$$

$$B(SS)\mathcal{L}_0^{-S} \xrightarrow{2} S(SS\mathcal{L}_0^{-S})(\mathcal{L}_0^{-S}(SS\mathcal{L}_0^{-S})) \subseteq S\mathcal{L}_0^{-S}(\mathcal{L}_0^{-S}\mathcal{L}_0^{-S}) \xrightarrow{*} (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}] \subseteq \mathcal{N}^{-S}.$$

  (Note: $\mathcal{L}_0^{-S}\mathcal{L}_0^{-S} \xrightarrow{*} (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}]$ was verified in (i) and (ii) above.) This way, we may verify our choice we have with:

$$n_1\, n_2 \;\in\; (SS)^*[B\mathcal{L}_0^{-S}]\, S \;\xrightarrow{*}\; (SS)^*[B\mathcal{L}_0^{-S}S] \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}] \;\subseteq\; \mathcal{N}^{-S} \qquad \text{and}$$

$$n_1\, n_2 \;\in\; (SS)^*[B(SS)]\, \mathcal{L}_0^{-S} \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[B(SS)\mathcal{L}_0^{-S}] \;\xrightarrow{*}\; (S\mathcal{L}_0^{-S})^*[\mathcal{N}^{-S}] \;\subseteq\; \mathcal{N}^{-S}.$$

- Suppose $n_2(n_1'n_2) \xrightarrow{*} \mathcal{H}_0^{-S}$ and $n_1'n_2 \notin \mathcal{L}_0^{-S}$. Then, $n_1' = S$ and $n_2 = SS$. In this case, we may verify:

$$n_1\, n_2 \;\in\; (SS)^*[BS]\,(SS) \;\xrightarrow{*}\; (S(SS))^*[SB(SSB)] \;\in\; (S\mathcal{L}_0^{-S})^*[S\mathcal{H}_0^{-S}\mathcal{L}_1^{-S}] \;\subseteq\; \mathcal{N}^{-S}.$$

In summary, $n_1 n_2 \xrightarrow{*} \mathcal{N}^{-S}$ for $n_1 \in \mathcal{H}_0 - S - S\mathcal{N}$, only if:

$$\begin{aligned} &n_1\, n_2 \in (SS)^*[S]\, \mathcal{L}_0^{-S}, && n_1\, n_2 \in (SS)^*[SBS]\, S, \\ &n_1\, n_2 = (SSS)\, B, && n_1\, n_2 \in (SS)^*[SB(SS)]\, (SS), \\ &n_1\, n_2 \in (SS)^*[S\mathcal{H}_0^{-S}]\, \mathcal{L}_0^{-S}, && n_1\, n_2 \in (SS)^*[B\mathcal{L}_0^{-S}]\, S, \\ &n_1\, n_2 \in (SS)^*[S\mathcal{L}_4^{-S}]\, S, && n_1\, n_2 \in (SS)^*[B(SS)]\, \mathcal{L}_0^{-S}, \qquad \text{or} \\ &n_1\, n_2 \in SS(SS)\, \mathcal{L}_1^{-S}, && n_1\, n_2 \in (SS)^*[BS]\, (SS). \end{aligned}$$

#2. Suppose $n_1 \in \mathcal{H}_1 \cap \mathcal{N}^{-SS}$ and $n_2 = SS$. Then, $n_1 \in (SS+B)^*[n_1']$ for some $n_1' \in S(SB + SS\mathcal{H}_0^{-SS})(S+SS)$. We will first show $n_1 n_2 \xrightarrow{*} (S\mathcal{N})^*[n_1' n_2]$. Then, we will show that $n_1' n_2 \xrightarrow{*} \mathcal{N}^{-S}$ is not possible, i.e., $n_1' n_2 S\uparrow$. From these results and Proposition R we can verify $n_1 n_2 \xrightarrow{*} \mathcal{N}^{-S}$ is impossible.
With Proposition R, we compute:

$$n_1\, n_2 = (SS+B)^*[n_1']\,(SS) \xrightarrow{*} (B + SS(SS))^*[n_1'(SS)].$$

Clearly $n_1'(SS) \in (S\mathcal{N})^*[n_1' n_2]$. Suppose $n \in (S\mathcal{N})^*[n_1' n_2]$. Then,

$$B\,n = B\big((S\mathcal{N})^*[n_1' n_2]\big) \subseteq (S\mathcal{N})^*[n_1' n_2] \qquad \text{and}$$

$$SS(SS)\,n \longrightarrow Sn(SSn) \subseteq (S\mathcal{N})^*[n] \subseteq (S\mathcal{N})^*[n_1' n_2].$$

Therefore, $n_1 n_2 \xrightarrow{*} (S\mathcal{N})^*[n_1' n_2]$.
We show in no case $n_1' n_2 \xrightarrow{*} \mathcal{N}^{-S}$ with the following:

$$\begin{aligned} n_1'\, n_2 \in{} & S(SB + SS\mathcal{H}_0^{-SS})(S+SS)\,(SS) \xrightarrow{*} \\ & S\big((SS)^*[SSS + SS(SS) + SSB + SB]\big)\big(S+SS\big)\big(SS\big) \xrightarrow{*} \\ & \big((SS)^*[SSS + SS(SS) + SSB + SB](SS)\big)\big((S+SS)(SS)\big) \xrightarrow{*} \\ & \big((B)^*[BB + B(SS(SS)) + B(B(SS)) + SB(SS)]\big)\big(B + SS(SS)\big). \end{aligned}$$

The left component in the final expression (not yet in normal form!) is a subset of $\mathcal{H}_0$. However, every application of a term in this left component with a term in the right component, B or $SS(SS)$, fails to match any $n_3 n_4 \in \mathcal{H}_0\mathcal{N}$ such that $n_3 n_4 \xrightarrow{*} \mathcal{N}^{-S}$ discussed before.

#3. Suppose $n_1 \in \mathcal{H}_1 \cap \mathcal{N}^{-S}$ and $n_2 = S$. Then, $n_1 = Sn_3n_4$ for some $n_3, n_4 \in \mathcal{N}$. Naturally, $n_1 n_2 = Sn_3n_4S \longrightarrow (n_3S)(n_4S)$ and $n_3S \xrightarrow{*} \mathcal{H}_0 + \mathcal{H}_1$. We reject $n_3S \xrightarrow{*} \mathcal{H}_1$ because if so, we would have to accept $n_1 n_2 \xrightarrow{*} \mathcal{H}_1(SS)$, but this resulting set was proven disjoint from $\langle \mathcal{N}^{-S} \rangle$ just above. Then, $n_3S \xrightarrow{*} \mathcal{H}_0$. Let $n_3', n_4' \in \mathcal{N}$ be such that $n_3S \xrightarrow{*} n_3'$ and $n_4S \xrightarrow{*} n_4'$. Then, $n_3' n_4' \in \mathcal{H}_0\mathcal{N}$. Therefore, either $n_3' n_4'$ matches one of the choices found in **#1**, when supposing "$n_1 \in \mathcal{H}_0$," or else $n_3' \in S\mathcal{N}$. For the first alternative, we extract $n_1 \in S\mathcal{L}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{K}_4S$ from the result at the end of **#1** ($\mathcal{K}_4 = (\mathcal{L}_2^{-S})^{-S}$ justifies the $S\mathcal{K}_4S$ part). This is because n_3' cannot be $B(SS)$, $SB(SS)$, nor in $\mathcal{L}_0 - \mathcal{L}_0^{-S}$, e.g., not in $(SS)^*[S\mathcal{L}_4^{-S}]$, and n_4' cannot be S, B, nor in $\mathcal{L}_1^{-S}$. The second alternative forces $n_3' = SS$, so $n_3 = S$ and $n_1 n_2 = SSn_4S \to SS(n_4S)$, which still requires $n_4S \xrightarrow{*} \mathcal{N}^{-S}$. Therefore, the second alternative only introduces the possibility of having any number of prefixes SS. However, the base expression must be an S-term given from the first alternative. Therefore, $n_1 n_2 \xrightarrow{*} \mathcal{N}^{-S}$ for $n_1 \in \mathcal{N}^{-S}$ and $n_2 = S$, only if:

$$n_1 \in (SS)^*[S\mathcal{L}_0^{-S}\mathcal{L}_0^{-S}] \qquad \text{or} \qquad n_1 \in (SS)^*[S\mathcal{K}_4S].$$

Let

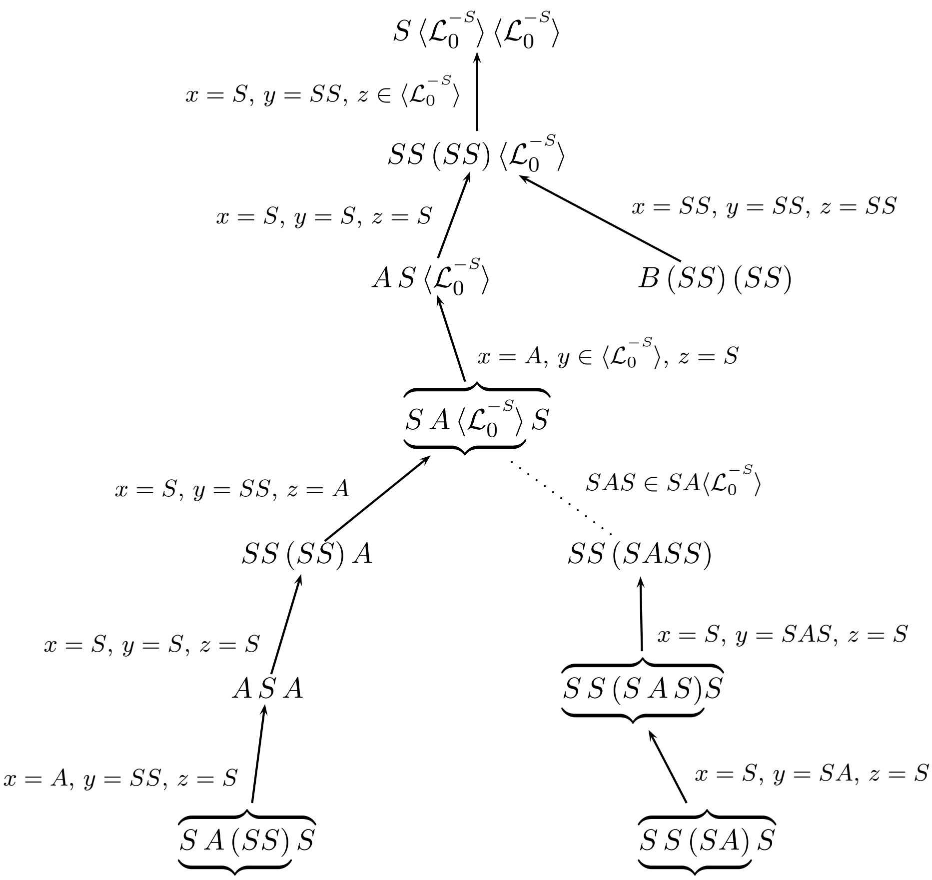
$$\mathcal{J}_1 \stackrel{\text{def}}{=} A = SSS\,, \qquad \mathcal{J}_6 \stackrel{\text{def}}{=} (SS)^*[B(SS)]\,,$$
$$\mathcal{J}_2 \stackrel{\text{def}}{=} (SS)^*[S\mathcal{H}_0^{-S}]\,, \qquad \mathcal{J}_7 \stackrel{\text{def}}{=} (SS)^*[B\mathcal{L}_0^{-S}]\,,$$
$$\mathcal{J}_3 \stackrel{\text{def}}{=} (SS)^*[S\mathcal{L}_4^{-S}]\,, \qquad \mathcal{J}_8 \stackrel{\text{def}}{=} (SS)^*[S\mathcal{L}_0^{-S}]\,,$$
$$\mathcal{J}_4 \stackrel{\text{def}}{=} SS(SS)\,, \qquad \mathcal{J}_9 \stackrel{\text{def}}{=} (SS)^*[S\mathcal{L}_0^{-S}S]\,, \qquad \text{and}$$
$$\mathcal{J}_5 \stackrel{\text{def}}{=} (SS)^*[SB(SS)]\,, \qquad \mathcal{J}_{10} \stackrel{\text{def}}{=} (SS)^*[S\mathcal{L}_0^{-S}\mathcal{L}_0^{-S} + S\mathcal{K}_4 S]\,.$$

We complete the grammar as follows:

$$\begin{array}{rlll}
\text{(xxi)} & \langle \mathcal{N}^{-S} \rangle & ::= & S \mid S\,\langle \mathcal{N} \rangle \mid S\,\langle \mathcal{H}_0^{-S} \rangle\,\langle \mathcal{L}_0^{-S} \rangle \mid S\,\langle \mathcal{L}_1^{-S} \rangle\,\langle \mathcal{H}_0^{-S} \rangle \mid S\,\langle \mathcal{L}_2^{-S} \rangle\,\langle \mathcal{L}_1^{-S} \rangle \mid S\,\langle \mathcal{L}_4^{-S} \rangle\,S \mid \\
& & & \langle \mathcal{K}_0 \rangle\,\langle \mathcal{L}_0^{-S} \rangle \mid \langle \mathcal{J}_1 \rangle\,B \mid \langle \mathcal{J}_2 \rangle\,\langle \mathcal{L}_0^{-S} \rangle \mid \langle \mathcal{J}_3 \rangle\,S \mid \langle \mathcal{J}_4 \rangle\,\langle \mathcal{L}_1^{-S} \rangle \mid \langle \mathcal{L}_1 \rangle\,S \mid \\
& & & \langle \mathcal{J}_5 \rangle\,(\,S\,S\,) \mid \langle \mathcal{J}_6 \rangle\,\langle \mathcal{L}_0^{-S} \rangle \mid \langle \mathcal{J}_7 \rangle\,S \mid \langle \mathcal{J}_9 \rangle\,S \mid \langle \mathcal{J}_{10} \rangle\,S \mid \langle \mathcal{L}_2^{-S} \rangle\,(\,S\,S\,) \mid \\
& & & S\,\langle \mathcal{L}_0^{-S} \rangle\,\langle \mathcal{N}^{-S} \rangle \\
\text{(xxii)} & \langle \mathcal{J}_1 \rangle & ::= & A \\
\text{(xxiii)} & \langle \mathcal{J}_2 \rangle & ::= & \langle \mathcal{K}_0 \rangle\,S \mid S\,\langle \mathcal{H}_0^{-S} \rangle \mid S\,S\,\langle \mathcal{J}_2 \rangle \\
\text{(xxiv)} & \langle \mathcal{J}_3 \rangle & ::= & \langle \mathcal{K}_0 \rangle\,S \mid S\,\langle \mathcal{L}_4^{-S} \rangle \mid S\,S\,\langle \mathcal{J}_3 \rangle \\
\text{(xxv)} & \langle \mathcal{J}_4 \rangle & ::= & S\,S\,(\,S\,S\,) \mid \langle \mathcal{J}_1 \rangle\,S \\
\text{(xxvi)} & \langle \mathcal{J}_5 \rangle & ::= & S\,B\,(\,S\,S\,) \mid S\,S\,\langle \mathcal{J}_5 \rangle \\
\text{(xxvii)} & \langle \mathcal{J}_6 \rangle & ::= & B\,(\,S\,S\,) \mid S\,S\,\langle \mathcal{J}_6 \rangle \\
\text{(xxviii)} & \langle \mathcal{J}_7 \rangle & ::= & \langle \mathcal{K}_4 \rangle\,S \mid B\,\langle \mathcal{L}_0^{-S} \rangle \mid S\,S\,\langle \mathcal{J}_7 \rangle \\
\text{(xxix)} & \langle \mathcal{J}_8 \rangle & ::= & S\,\langle \mathcal{L}_0^{-S} \rangle \mid S\,S\,\langle \mathcal{J}_8 \rangle \\
\text{(xxx)} & \langle \mathcal{J}_9 \rangle & ::= & \langle \mathcal{J}_8 \rangle\,S \mid S\,S\,\langle \mathcal{J}_9 \rangle \\
\text{(xxxi)} & \langle \mathcal{J}_{10} \rangle & ::= & S\,S\,\langle \mathcal{J}_{10} \rangle \mid S\,\langle \mathcal{L}_0^{-S} \rangle\,\langle \mathcal{L}_0^{-S} \rangle \mid SS\,(SS)\,\langle \mathcal{L}_0^{-S} \rangle \mid A\,S\,\langle \mathcal{L}_0^{-S} \rangle \mid B\,(SS)\,(SS) \mid \langle \mathcal{J}_{11} \rangle\,S \\
\text{(xxxii)} & \langle \mathcal{J}_{11} \rangle & ::= & S\,S\,\langle \mathcal{J}_{11} \rangle \mid S\,\langle \mathcal{K}_4 \rangle \mid S\,A\,\langle \mathcal{L}_0^{-S} \rangle \mid S\,S\,(SS)\,A \mid A\,S\,A \mid \langle \mathcal{J}_{12} \rangle\,S \\
\text{(xxxiii)} & \langle \mathcal{J}_{12} \rangle & ::= & S\,S\,\langle \mathcal{J}_{12} \rangle \mid S\,S\,(SA) \mid S\,A\,(SS) \\
\text{(xxxiv)} & A & ::= & S\,S\,S
\end{array}$$

As we can see, for every right rule part of the form $X\,S$, where X is a non-terminal, the rule $X ::= SS\,X$ must be included in the set of rules for X.

The list of predecessors for $\mathcal{J}_{10}$ was obtained with an analysis like the one shown in the following diagram (the $(SS)^*[S\mathcal{K}_4S]$ case is not shown in the diagram; such terms are immediately put in $\langle \mathcal{J}_{11} \rangle$). Terms shown overbraced and underbraced in the diagram are at the boundaries between $\langle \mathcal{J}_{10} \rangle$ and $\langle \mathcal{J}_{11} \rangle$ ($S\,A\,\langle \mathcal{L}_0^{-S} \rangle\,S$ and $S\,A\,\langle \mathcal{L}_0^{-S} \rangle$ respectively) or between $\langle \mathcal{J}_{11} \rangle$ and $\langle \mathcal{J}_{12} \rangle$ ($S\,A\,(SS)\,S$ and $S\,A\,(SS)$ respectively, $S\,S\,(SA)\,S$ and $S\,S\,(SA)$ respectively).


$S\,\langle\mathcal{L}_0^{-S}\rangle\,\langle\mathcal{L}_0^{-S}\rangle$
$x = S,\ y = SS,\ z \in \langle\mathcal{L}_0^{-S}\rangle$
$SS\,(SS)\,\langle\mathcal{L}_0^{-S}\rangle$
$x = S,\ y = S,\ z = S$
$x = SS,\ y = SS,\ z = SS$
$A\,S\,\langle\mathcal{L}_0^{-S}\rangle$
$B\,(SS)\,(SS)$
$x = A,\ y \in \langle\mathcal{L}_0^{-S}\rangle,\ z = S$
$S\,A\,\langle\mathcal{L}_0^{-S}\rangle\,S$
$x = S,\ y = SS,\ z = A$
$SAS \in SA\langle\mathcal{L}_0^{-S}\rangle$
$SS\,(SS)\,A$
$SS\,(SASS)$
$x = S,\ y = S,\ z = S$
$x = S,\ y = SAS,\ z = S$
$A\,S\,A$
$S\,S\,(S\,A\,S)S$
$x = A,\ y = SS,\ z = S$
$x = S,\ y = SA,\ z = S$
$S\,A\,(SS)\,S$
$S\,S\,(SA)\,S$